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BIOMETRIKA

KARL PEARSON

An Appreciation of some Aspects of his Life and Work

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Part I: 1857–1906*

FOREWORD

To set before the readers of this Journal an adequate account of the life and work of the man who, since its founding, has been so intimately associated with every aspect of its editorship and production would be no easy task. It is the more difficult when we are concerned with one whose influence has been felt far beyond the pages of *Biometrika*, with a great figure who, for more than forty years, has taken a place in the forefront of the intellectual forces of science.

To do full justice to the output of Karl Pearson's fertile brain would demand a prolonged and careful study of the immense range of his published work; nor indeed has the time yet come when any final summary of the value of his contribution to science can be made. But the passing of the last of the three founders of *Biometrika* marks inevitably the closing of an epoch, and it falls to the lot of him who has been asked to assume the responsibility of editorship to attempt a salute fitting to that close.

With the lapse of thirty-five years the conditions which led to the origin of the Biometric School are largely altered; this evolution indeed has been reflected in the changing character of the contributions to these pages. No spearhead of enthusiastic workers is now required to lead a fight for the recognition of the place of mathematics in the biological field. The value of statistical method has been almost universally accepted among biologists, and tools which trace their origin to Pearson's workshop are applied along widely spreading lines of research investigation. Nevertheless in this world of ever increasing specialisation there is perhaps some danger of losing touch with the simple yet bold objectives which, in the launching of a new branch of science, the leaders of thought and action set before themselves. To portray these objectives, and to catch something of the spirit in which the master we have lost followed them for so long in single-eyed pursuit, must be the main task of the writer of this memoir.

* The second part of this memoir will appear shortly in *Biometrika*, and the two parts will afterwards be re-issued in book form.

In that delightful autobiographical sketch (1)* which he gave us at the dinner held in his honour in April 1934, Pearson told us that we should find an explanation of his life in a combination of two inherited characteristics, "a capacity for hard work and a capacity for roving into other people's preserves." The importance of these two contributions to his life history need not be disputed, yet we should miss the essential characteristic which made him a leader of scientific thought if we accepted this self-explanation as sufficient. On the first page of the Memoir (2) of his friend and colleague, W. F. R. Weldon, we find the following sentences:

> "But science, no less than theology or philosophy, is the field for personal influence, for the creation of enthusiasm, and for the establishment of ideals of self-discipline and self-development. No man becomes great in science from the mere force of intellect, unguided and unaccompanied by what really amounts to moral force. Behind the intellectual capacity there is the devotion to truth, the deep sympathy with nature, and the determination to sacrifice all minor matters to one great end."

This moral force, this "creed of life," as he described it, which makes a man "serve science from love as men in great religious epochs have served the Church," was developed in Pearson to a marked degree. It was no doubt a characteristic of the Victorian age, of a period when scientific thought, bursting free from the traditions of an orthodox Christianity which it had learnt in its youth, felt still a pressing need for some new philosophy of life to replace the old. In the life of Karl Pearson we may trace all the signs of a struggle for freedom, of a period of uncertainty and trial, of the development of a new faith and of the blending of this faith with his outlook on science. It follows that although in this short memoir we must be concerned largely with the biometrician and statistician, to understand the man of science we must dip first into his early years.

Early Years

Karl Pearson was born in London on 27 March, 1857†. On both sides he was of Yorkshire descent. The Pearsons were of yeoman stock, and during the last forty years of his life Karl spent many summer vacations among the dales of the North Riding. Here he experienced that satisfaction, that sense of balance and stability, which a man feels among his own folk, and he loved to mix more serious work at statistics with walks along the tracks and bridle-paths between the dale-side farms where his Quaker ancestors had lived and died; and to pick up here and there from the memories of the old people, from wills and church records, something of the history of his stock. On his mother's side there were seamen, master-mariners who sailed their own ships from Hull, but he never seems to have felt strongly the call of the sea.

William Pearson, his father, was a barrister and Q.C. He was a man of great ability, a man with an exceptional power for work, who before he retired was

* This and subsequent reference numbers in brackets relate to the short bibliographical list of Pearson's writings given on pp. 242–244 below.

† He had one brother, Arthur, born in 1855, and a sister, Amy, born in 1859.

a leader in the Chancery Courts. He had too, like his son, a keen interest in historical research, and left at his death a library of books and many unpublished notes dealing with such subjects as the Anglo-Saxon Chronicle, Domesday Book, the history of Yorkshire, etc. Karl Pearson, in the speech I have referred to (1), spoke of him as follows: "During the legal terms, winter and summer, he was up at 4 a.m. to read his briefs and prepare his speeches for Court. Home at 7 p.m., dinner followed and bed at 9 p.m. Only in the vacations did we really see him; then he was shooting, fishing, sailing with a like energy which astonished me even as an active boy." In the early contact between William and Karl there lies perhaps much that would throw light on some of the characteristics of the son in later life, but the material to aid us here is scanty.

The younger Pearson was sent to University College School, London, in 1866; the early portrait facing p. 202 shows him at about this time. He seems to have been delicate and was possibly restrained by somewhat anxious parents from activities with other boys. Indeed in 1873, at the age of sixteen, he was withdrawn from school for reasons of health and spent the next year with a private tutor at Hitchin. His own words taken from a recent reminiscence of Cambridge days (3) may be quoted again; speaking of this private tutor he says:

"He was not a bad teacher but unable to manage a party of young men who did not want to learn and whose follies easily developed into vices. Perhaps the most useful knowledge I acquired from him was an appreciation of Dynamics, and from his pupils an acquaintance with dog-breeding which has been useful to me in after life. At the end of the year I persuaded my father to let me go up to Cambridge and work as a 'beast' under Routh....[On the first morning] at seven o'clock I went and found another 'beast' there, 'Josh' Conway (now Lord Conway of Allington). Routh said to us: 'You have a year before entering college; we will devote it to reading subjects not of first-class importance for the Tripos,' and he started straight off to lecture on the theory of elasticity. Conway dropped away after a term, and Routh took me alone at seven o'clock in the morning throughout the year, introducing me to Lamé's works and papers by other writers."

We see here pointers towards the future, as no doubt the writer with his strong sense of historical development intended us to do: Dynamics and the illustrative material of *The Grammar of Science*; Lamé and the completion of Todhunter's *History of the Theory of Elasticity*; dog-breeding and that experimental investigation in the crossing of albino Pekinese and black Pomeranians, which also provided the dogs that meant so much to him at home in later years.

In 1875 he obtained a scholarship at King's College, Cambridge, being placed second on the list. He has left us an account (3), mellowed by the passing of years, of what Cambridge gave him: pleasure in friendships, pleasure in fights, pleasure in the coaches' teaching, pleasure in searching for new lights as well in mathematics as in philosophy and religion. In that small community at King's, containing at that time between thirty and forty members in all, there was opportunity for close contact between undergraduates and dons which led to the development of personal friendships having a lasting influence on many of the younger men. Pearson has

told us (3) of Oscar Browning, who introduced him to Rousseau, Goethe and Italy; of G. W. Prothero, "a man who could hold the balance...between the mediaeval provost and hustling undergraduate," and finally of Henry Bradshaw, the University Librarian, with whom a friendship of much import was to ripen in post-graduate years.

He gives us an amusing account in the same place of the Smith's Prize Examination held on four successive days at the houses of the four examiners, Stokes, Clerk-Maxwell, Cayley and Todhunter. Pearson failed to become Prizeman, but one of his answers to a question of Todhunter's, bound up by the latter with the manuscript of his unfinished *History of the Theory of Elasticity* with the comment "This proof is better than De St Venant's," led afterwards to Pearson's appointment by the Syndics of the Cambridge University Press to finish and edit that work.

We see him also appearing already as the determined fighter against what he regarded as misdirected authority:

"I went to the tutor and said, 'I am not going to attend any more divinity lectures.' He looked aghast. 'It is an inexorable rule of the College.' 'Then I am going to another college.' 'It is unheard of. We shall not give you a *bene decessit*.' 'Mr Ferrers of Caius has already agreed to accept me without a *bene decessit*.' 'This matter must come before the College Council.' It did come, and after a hard fight compulsory divinity lectures were abolished*."

But if he resented this compulsion, he was still intensely interested in the spirit, the doctrine and the history of religion. It was as though, before he cast loose from his old moorings, he was searching in a critical spirit to find what was of value to take with him. The Passion-Play (4), which he wrote and published anonymously a few years later, shows the depth of thought and emotion which accompanied this voyaging. It was in undergraduate days also that he was first attracted by the work of Spinoza, whose philosophy influenced to a considerable extent the direction of this journey. Almost at the end of his life we find him writing (3): "I can only say that till this day I think Spinoza the sole philosopher who provides a conception of the Deity in the least compatible with scientific knowledge."

Of contemporary friendships we may note in particular that with Robert Parker, afterwards, as a judge of the Court of Appeal in Ordinary, Lord Parker of Waddington, and with W. H. Macaulay, later mathematical lecturer, Tutor and Vice-Provost of King's. From these we may imagine he received counsel and sympathy during the coming wander-years; help too they gave him in the revision of his writings, and in trying to keep his "scientific radicalism within moderate and reasonable bounds."

In 1879 Pearson took his degree with Mathematical Honours, being Third Wrangler in the Tripos of that year. Almost at once he left for Germany.

* Long afterwards Pearson might remember only the "pleasure in fights," but from letters written by his father while the issue was still in doubt we can gather that the boy, whose emotions were roused in the battle, was also depressed and troubled by the incident.

1879–1884

The next five years represent a phase of great interest and importance in Pearson's development. The Fellowship of King's College, which was conferred on 5 April, 1880, assured his financial independence for a number of years, and he made good use of this opportunity to roam at will. It was a period for him of much reading, of intense intellectual activity with an urge for self-expression, whether in letters, poems, critical reviews, essays or lectures. Out of an extensive printed record we can trace, slowly emerging, the later man of science with his characteristic creed of life. In the space of the present memoir, I can do no more than attempt to follow some of his activities, to refer to some of the influences at work, and finally, with a few extracts picked from his writings, to outline what appears to me to have been the philosophy of life that he was building up. At best, it must be but an imperfect sketch of the man.

After taking his degree, no doubt his father's profession of the Law was in Pearson's mind. He took rooms in November 1880, at 2, Harcourt Buildings, Inner Temple, he read in Chambers at Lincoln's Inn and was called to the Bar in 1881. But his legal studies formed only one of many activities. I have mentioned that he went to Germany soon after taking his degree; he had already made at least one visit to the Rhine with his parents in earlier days. Now, in 1879 and 1880, he was at Heidelberg and Berlin; his original objective in this move is not clear. In his Dinner Speech he told us in paradoxical vein (1): "In Heidelberg I studied Physics under Quincke, but also Metaphysics under Kuno Fischer. In Berlin I studied Roman Law under Bruns and Mommsen, but attended the lectures of Du Bois Reymond on Darwinism." One thing is, however, certain; he soon became interested in the study of German folklore, in the history of the Reformation and in the characters of the German Humanists and of Luther. During the next few years, after he had fixed his headquarters in London, he was frequently back in Germany; he had found a quiet spot in the Black Forest at Saig to which he returned often between 1880 and 1890 to read and write. Here, speaking the language fluently, he got into close touch with the people of the land, as he did at a later period in his ancestral Yorkshire dales.

It was in this field of German History that he made his first serious attempt at original research, and from that contact with facts obtained the balance that was perhaps needed during a time of emotional development. Henry Bradshaw, the Cambridge University Librarian, must have influenced him much at this time, and taught him the meaning of thoroughness and patience in research. Pearson has paid this tribute to the help which he gave (1):

"Bradshaw I had known in my undergraduate days. He was the ideal librarian, but something greater—the guide of the young and foolish. After my return from Germany during 1880 to 1886 his influence as a scholar on me grew greater and greater. His visits to my chambers in the Inner Temple and our joint visits to the Long Room of the British Museum were epochs. He did not directly teach me to concentrate, he was willing, like a good librarian, to accompany me into any field, but he showed me

what the essentials of true workmanship must be. The following words always come to my mind when I think of Henry Bradshaw:

> The song I sang, old Languet had me taught:
> Languet the shepherd best swift Istar knew,
> For clerkly reed, and hating what is naught,
> For faithful heart, clean hands and mouth as true.
> With his sweet skill my skilless youth he drew
> To have a feeling taste of Him that sits
> Beyond the Heavens, far more beyond our wits.
>
> SIDNEY'S *Arcadia*."

The depth of that friendship and the extent to which Bradshaw stood, perhaps, as a symbol of what Cambridge meant to Pearson, are shown by the wound left by his death in 1886. Writing of Pearson long afterwards, a friend has said: "He was in Cambridge frequently when I was in residence, about 1884–86, and was then helped by Henry Bradshaw in collecting material for *Die Fronica* (5). But after Bradshaw's death, he came very seldom. He said on the day of the funeral that he thought he could never come again."

The results of this historical research appeared in many forms. In the spring of 1882 Pearson gave a course of ten lectures at the South Place Institute* on "German Social Life and Thought," the object of which was "to trace the rise and fall of successive German Ideals from the earliest times up to A.D. 1500, which period embraces the growth and decay of the Principle of *Unity* in Germany, as embodied in the Holy Roman Empire and the Holy Catholic Church." Later in the same year, in connection with University Extension Teaching, he gave twelve lectures at Hampstead on "German Social Life and Thought in the Sixteenth Century," a course dealing largely with the period of the Reformation. These were followed by articles on "Humanism in Germany" (1883), "The Kingdom of God in Münster" and "Martin Luther: his Influence on the Material and Intellectual Welfare of Germany" (1884), all of which were later republished in *The Ethic of Freethought* (6) in 1888. At this time also he was collecting material on the German Passion-Play, put together and published some twelve years afterwards in *The Chances of Death* (7).

His study of Spinoza and Spinozan literature continued, and between 1880 and 1883 he had written three long reviews on the subject in *The Cambridge Review*, and an article on Maimonides and Spinoza in *Mind*†.

Besides this more substantial work, a flow of letters, articles and reviews appeared in *The Academy* and *The Athenaeum* and elsewhere, many concerned with Luther, the four hundredth anniversary of whose birth was celebrated in 1883. In these we find him giving and receiving blows with vigour; he criticised

* The Institute was, I believe, run at this date by the South Place Ethical Society. The Sunday Meetings, which were a feature of the organisation, combined good music and readings of prose or poetry with an address. One who was present at Pearson's address on "The Enthusiasm of the Market-place and of the Study" in 1885 recalls that this was preceded by the reading of Browning's "The Grammarian's Funeral."

† This was afterwards republished in *The Ethic of Freethought*.

a Luther Exhibition in the Grenville Library of the British Museum, saying that "the introduction and descriptions of the catalogue are really a slur on English Scholarship." He was told in reply that "there is but a step between hyper-criticism and hyper-nonsense." He attacked Matthew Arnold for speaking of Luther as "another example of lucidity," retorting that Luther was the "Cleon of Wittenburg." He criticised another writer for the misspelling of this last word, and was described editorially as that "'terrible person' (and very talented writer) K.P."

These literary duels bring before us the picture of a young man, clever, self-confident, with an unusually clear grasp of his subject, who, where he disagrees, is often too eager to rush in and try to set the world aright. He needed, and he found, the hand of an older man to curb his ardour. There is a letter of Bradshaw's, which has already been published*, dealing with the Museum controversy referred to above:

"I have not the slightest wish to defend the Museum ignorance. But...when a man who might by his own deeper knowledge help to make such an exhibition very much more interesting and instructive wastes his energies in writing to the *Athenaeum* as you do, it naturally produces the impression that his main object is to let the world see how much more he knows of the subject than the idiots to whose care he says these treasures are entrusted. Those who know you know also that that is not the object you have in view, but it is a pardonable inference for ordinary people to draw. Everything you write about this shows such an extraordinary absence of wisdom (by which I don't mean knowledge or cleverness, both of which are abundantly shown); but what pains me is that it is all so diametrically opposed to the whole spirit of the paper of yours on Free Thought, which you sent me; a paper which, as you know, delighted me more than any words of mine can express. You have such a fund of strong gentleness in you, that it is most cruel to see the immense power that this gentleness possesses simply wasted by a perversity as noxious as anything to be found even on the staff of the British Museum. I hope you will come on Tuesday, and you can sit on me to your heart's content."

The fact that Pearson put this letter at Prothero's disposal is proof, I think, that he recognised the wisdom and value of the criticism it contained.

That the young Pearson should have been a socialist was almost inevitable, since the philosophy of life that he was developing in these days linked the moral with the social, the social with the welfare and happiness of the many. It is easy, too, to see the connection between this trend of thought and his study of early German social life during "a period in which," he writes†, "the forces tending to revolutionise society were in many respects akin to those we find in action at the present time." Of his own socialistic activities at this period he has told us (1) that he lectured "on Lassalle and Marx on Sundays at revolutionary clubs round Soho. Indeed I contributed to the Socialist Song Book hymns which I believe are still chanted‡."

* *A Memoir of Henry Bradshaw*, by G. W. Prothero (Kegan Paul, Trench & Co. 1888), p. 399.

† Preface to *The Ethic of Freethought*.

‡ Some of his articles and songs written between 1880 and 1883 will be found in the pages of *The Cambridge Review*.

Among all these activities, his mathematical talent was not lying entirely fallow. During the first two terms of 1881 he deputised for W. H. Drew, Professor of Mathematics at King's College, London, taking the senior mathematical teaching. It was in 1882, according to his friend W. H. Macaulay, that he was engaged on his first considerable piece of mathematical work, "a theory of pulsating spheres in a fluid, forming an Atomic Theory,...a thing in spherical harmonics of the Clerk-Maxwell type." This work, or part of it, was probably not published till 1887. In 1883 two papers were printed "On the Motion of Spherical and Ellipsoidal Bodies in Fluid Media" (8), and another "Note on Twists in an Infinite Elastic Solid" (9). It was a natural step from such research to the completion of Clifford's *Common Sense of the Exact Sciences* (10) and Todhunter's *History of the Theory of Elasticity* (11). We see from the subject-matter of these papers how Pearson's mind was already at work puzzling over the laws of the physical universe, which in the terms of *The Grammar of Science* (12) describe the "how" rather than the "why."

These were some of the outward activities during this period; it is necessary now to attempt the more difficult task of tracing the movement beneath the surface. In 1880 there appeared, from the hand of an author writing as "Loki," a small book published by C. Kegan Paul and Co., entitled *The New Werther* (13). This was Pearson's first published work; it is written in the form of letters from a young man, Arthur, wandering in Germany, to Ethel, his betrothed, from whom he is parted for a time. In a short preface he explains his purpose:

"There is only one excuse for publishing the following letters, namely, they truly image the mind of him who has written them, and therefore necessarily to some extent the minds of the children of his generation, who are passing through a like struggle. They are beneath the notice of the critic, and appeal only to the sympathies of that young life which is cast creedless on the weary waters of nineteenth-century thought."

The letter-writer, Arthur, is searching for such a creed of life; he turns to many of the philosophies and religions of the past and present, and to the history of science. He builds up only to cast away. The letters also contain a number of poems and some vivid descriptions of the countryside. It must suffice to give a few quotations which will show how, in these early days, bits of the philosophy of *The Ethic of Freethought* and *The Grammar of Science* were being flung up to view in a youthful attempt to storm the universe:

"Do not expect to hear of any system or method on my part; time seems too short and the world too vast for that! I rush from science to philosophy, and from philosophy to our old friends the poets; and then, over-wearied by too much idealism, I fancy I become practical in returning to science. Have you ever attempted to conceive all there is in the world worth knowing—that not one subject in the universe is unworthy of study? The giants of literature, the mysteries of many-dimensioned space, the attempts of Boltzmann and Crookes to penetrate Nature's very laboratory, the Kantian theory of the universe, and the latest discoveries in embryology, with their wonderful tales of the development of life—what an immensity beyond our grasp! As I walk over the hills here, what a feeling of shame comes over me that I know nothing of botany or geology; that though I may gaze with loving eye on flower and rock, I am ignorant

of the life-history contained in the one and the world-history in the other! But my head swims; I feel our nature is too small, our powers are nought. Happy is the man who not only can say, 'I know nothing,' but can content himself with that nothing."

"It is to this problem...the union of the divine and human, the feeling of the existence of which led to its symbolification in the man-god, Christ—that I wish to devote my life. For this purpose I must study what the great world-minds have thought, and learn the motives of their action."

But the works of the philosophers did not satisfy:

"Failures, false hopes and disappointments innumerable! And yet not wholly useless, for I have learnt that man, before he reasons, must know more of Nature's ways; that only when he has found her more general laws will he be able to explain why they *are*—their intellectual origin."

It is in scientific research that hope seems to lie:

"Mankind seems on the verge of a new and glorious discovery. What Newton did to simplify the various planetary motions must now be done to unite in one whole the various isolated theories of mathematical physics. Again, what boundless fields remain to be explored in pure mathematics! Every identity in symbols is symbolic of a real truth, had man but power to interpret it. Give the symbols but consistent natural meanings, and a law of Nature, deep perhaps in its signification, is the result."

But if Karl was to pass on to a maturer balanced faith, to Arthur, the writer of the letters, there was to be no solution:

"Religion once tyrannised over the world. Science has killed Religion; but, instead of setting up a republic of thought, has instituted a worse tyranny in its place, the oligarchy of scientific specialists, who expect mankind at large to accept on the ground of authority whatever they choose to proclaim as truth!"

"Man can only live by faith, and must have faith to live. There is a craving in his nature for the mystic, whose existence Science ignores—nay, would root up. Whether anything can be found...to satisfy this yearning after the Unknown, I do not dare to assert; but of one thing I am sure—science will not and cannot. Religion in the twelfth century may have satisfied it, but has long ceased to do so...."

There was one sympathetic and understanding review of this work in *The Cambridge Review* of 10 November, 1880, written perhaps by an older friend of the author.

A publication which received more attention and some hostile criticism was another anonymous work entitled *The Trinity; a Nineteenth Century Passion-Play* (4), published in 1882. A great deal of historical study, of thought and emotional energy must have gone into the production of this work, which is written largely in verse. To the author and to those who knew him closely, there was much in it having a symbolic meaning centring in the interplay of duty, love and reason. But the critics, troubled primarily by the lack of orthodoxy, could hardly be expected to detect this. In the preface it is written,

"As to the scope of the book—its λόγος—that must speak for itself. Modern science and modern culture are freeing us from the old theological shackles; let them take heed that in destroying a human divinity they do not forget a divine humanity."

The second portrait on the opposite page, taken at Lenzkirch in Germany in 1882, shows the Karl Pearson of this period.

By 1883 the troubles of "Arthur" seem to be on their way to solution. A lecture given at South Place Institute on 6 March of that year, entitled "The Ethic of Freethought," was the first of a series delivered during the next few years, in which we can find, gradually taking shape, that faith of a scientist which formed the basic philosophy of Pearson's life. These lectures, given between 1883 and 1887, and most of them printed separately, were published together in 1888 under the common title of *The Ethic of Freethought* (6). There is a unity of purpose and feeling throughout the whole, and though several fall, in date of delivery, beyond the period of this section of the memoir, it will be best to consider them together.

The first lecture, which gave the title to the volume, is essentially that of a young man filled with the excitement of life, who in rejecting the dogmas of the past is perhaps, in spite of his own disclaimer, somewhat dogmatic in asserting the truth of his own philosophy. But that philosophy will bear some study.

The great problem which early forced itself on the attention of man was the "Eternal Why." Why am I here? What relation do I, a part, bear to the whole, to the sum of all things material and spiritual? What connection has the finite with the infinite? In answer to this questioning grew up religion. But the great danger of most existing systems of religion is this: "not content with our real knowledge of the relation of the finite to the infinite, they slur over our past ignorance by the help of the imagination. *Myth* supplies the place of true knowledge...."

Science and philosophy are continually presenting us with broader views of the relation of man to nature, of individual thought to abstract thought, and every concrete religion ought therefore to be in a state of development. But they are hardly so, they persist in explaining by myth, by remaining dogmatic. "The rejection of all myth explanation, the frank acceptance of all ascertained truths with regard to the relation of the finite to the infinite, is what I term *freethought* or true religious knowledge. In other words, the freethinker, in my sense of the term, possesses more real religion...than any believer in myth; his very knowledge makes him in the highest sense of the words a religious man." Freethought is difficult to attain, it is perhaps an ideal rather than an actuality: "To become a freethinker, it is not sufficient to throw off all forms of dogmatism, still less to attack them with coarse satire; this is but negative action. The true freethinker must be in the possession of the highest knowledge of his day; he must stand on the slope of his century and mark what the past has achieved, what the present is achieving; still better if he himself is working for the increase of human knowledge or for its spread among his fellows—such a man may truly be termed the high priest of freethought."

"Like the true man of science, the freethinker must never be ashamed to say: Here I am ignorant, this I cannot explain." Yet freethought must not be identified

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with science. It "is knowledge of the relation of the finite to the infinite, and science, in so far as it explains the position of the individual with regard to the whole, is a very important element, but not the totality, of such knowledge." "The greatest poets, philosophers, and naturalists, men such as Goethe, Spinoza and Darwin, have all been freethinkers; they strove, regardless of dogmatic belief and armed with the highest knowledge and thought of their time, to cast light on the one great problem of life."

It lies within the power of but few individuals to add any marked contribution to the stock of gradually accumulating knowledge; what is added in a generation is insignificant compared to what is handed down to us from the past. "Every freethinker, then, owes an intense debt of gratitude to the past; he is necessarily full of reverence for the men who have preceded him; their struggles, their failures and their successes, taken as a whole, have given him the great mass of his knowledge. Hence it is that he feels sympathy even with the very failures, the false steps of the men of the past. He never forgets what he owes to every stage of past mental development."

An important part of the mission of freethought, in which its essentially religious character appears, lies in the discovery of new truth. In this pursuit of knowledge lies the highest pleasure of which the human mind is capable; the intense delight in discovery which must have been experienced by a Kepler or a Newton.

But what is the nature of this truth that must be sought? It is essentially law, and this law is related to human thought. "Many of our so called laws are merely empirical laws, the result of observation; but the progress of knowledge seems to me to point to a far distant time when all the finite things of the universe shall be shown to be united by law, and that law itself to be the only possible law which thought can conceive." Again, "we have to look upon the universe as one vast intellectual process, every fact corresponds to a conception, and every succession of facts to an inevitable sequence of conceptions; as thought progresses in logical order of intellect only, so only does fact. The law of the one is identical with the law of the other."

And so we are led to the final conclusion; in the very nature of the problem of life "lies a strange inexpressible pleasure; it is the apparently finite mind of man which itself rules the infinite; it is human thought which dictates the laws of the universe; only what man can *think* can possibly *be*....Freethought, in making the freethinker master of his own reason, renders him lord of the world. That seems to me the endless joy of the freethinker's faith. It is a real and a living faith which, creative, sympathetic, and above all enthusiastic, is destined to be the creed of the future."

It would be an interesting task, but one beyond the scope of this memoir, to examine how this statement of faith stands out against the background of thought of the early 1880's and to trace the sources which contributed to its form. It is worth while however noting one or two points which are, I think, suggestive.

We see here the origin of that interest, akin to reverence, for the great thinkers of the past which expressed itself later in Pearson's extensive collection of portraits and quotations in the form of mottoes which hang now on the walls of his old Department at University College. Many too will remember the sympathetic, almost affectionate, treatment of the past statisticians who formed the subjects of his lectures on the History of Statistics given between 1920 and 1930.

Those who may have read *The Grammar of Science* will note also that Pearson's views on the relation of law to thought were not yet expressed with that clearness which appeared at a later date. Indeed a full study of the text seems to show certain points of inconsistency in the presentation.

We must also recognise that in the creed of the freethinker there is a danger of intellectual intolerance, a possibility that, in the words of a critical reviewer, freethought might become "what Mr Karl Pearson thinks." This tendency towards dogmatism, at which "Arthur" of *The New Werther* had despaired, can be traced in some of Pearson's more controversial scientific work. As a teacher he could warn others of the danger, but as a fighter roused in the heat of a battle for what he believed was the truth, he could not himself altogether avoid it. Yet if we criticise, we must also ask: Without this confident belief in his own judgment, could he ever have achieved so much?

The later lectures of the series add a depth to the philosophy of life which was forming in 1883. In "The Enthusiasm of the Market-place and of the Study" (29 November, 1885) we find emphasised the need for patient search after truth, whether in the study of comparative history or in the complex questions of biology. The social problems of the day cannot be solved by bringing them into the market-place, with all its passion and prejudice. "Morality is not the following of a social impulse, but a habit of action based upon character, a habit moulded by that knowledge of truth which must become an integral part of our being." Examples of this are drawn from the past. "But I can give you a still more striking instance of how the men of the study have based morality upon knowledge. I refer to that little band of real workers, to the Humanists of the early sixteenth century. Men like Erasmus, Sebastian Brant, and Conrad Muth were working for a real reformation of the German people on the basis of education, of knowledge, of that progress which alone is sure, because it is based on reason. These men one and all identified immorality with ignorance; the immoral man with the fool. Feared on the one side by the monks, abused on the other by the Lutherans, they were asked: 'Who will absolve you bad Christians?' 'Study,' they replied, 'and Knowledge.'"

Passages such as these help us, I think, to understand many later occurrences in Pearson's life. We can follow why other men of science, whether his opponents in controversy or not, were impressed by the sincerity of his search for truth*. We

* Thus William Bateson at the beginning of the long controversy, which will be referred to below, wrote to Pearson in a letter of 13 February, 1902: "I respect you as an honest man, and perhaps the ablest and hardest worker I have met, and I am determined not to take up a quarrel with you if I can help it. I have thought for a long time that you are probably the only Englishman I know at this moment whose first thought is to get at the truth in these problems..."

can understand the fervour with which he chastened those who, in dealing with some great social problem, were, he thought, using an appeal to the emotions under the false garb of a scientific argument; they were not fools, they were knaves!

In the same lecture there are further references to the importance of law: "Understand what I mean by the end of education. I do not mean mere knowledge of scientific or historic facts; but these facts co-ordinated into laws, and these laws made so much a mode of thought that they are the received rules of human action."

We also find a reference to that belief in the essential gradualness of any lasting changes, drawn from Darwin's work, which was characteristic of Pearson's views on evolution. "So it is," he wrote, "that the scarcely perceptible influence of enthusiasm of the study may with the centuries achieve more than all the strong eloquence of the market-place."

In "Matter and Soul," a lecture delivered before the Sunday Lecture Society at St George's Hall on 6 December, 1885, exactly a week after the previous lecture, we find the development of ideas leading towards the teaching of *The Grammar of Science* and see emphasised that combination in Pearson of the philosopher and thinker with the man of science.

The two lectures on Socialism ("The Moral Basis of Socialism" delivered at some date before 1887 and "Socialism in Theory and Practice," 1884) help to explain further the idealistic creed of the young historian and scientist to whom the moral was the social. Let me quote two paragraphs:

"Man, in judging of conduct, is concerned only with the present life; he has to make it as full and as joyous as he is able, and to do this consciously and scientifically with all the knowledge of the present, and all the experience of the past, pressed into his service. Not from fear of hell, not from hope of heaven, from no love of a tortured man-god, but solely for the sake of the society of which we are members, and the welfare of which is our welfare—for the sake of our fellow men—we act morally, that is, socially."

"Socialism arises from the recognition (1) that the sole aim of mankind is happiness in this life, and (2) that the course of evolution, and the struggle of group against group, have produced a strong social instinct in mankind, so that, directly and indirectly, the pleasure of the individual lies in forwarding the prosperity of the society of which he is a member."

Two other points in these lectures may be noted. Reference is made to Malthus and to the problems of sex and of population which "will come more and more into the foreground," and which it is "of really urgent importance to discuss earnestly, scientifically and from every possible standpoint." There is also another reference to the doctrine of "gradualness" which must have weighed with him in his later criticism of mutation theory. "You may accept it as a primary law of history, *that no great change ever occurs with a leap*; no great social reconstruction, which will permanently benefit any class of the community, is ever brought about by a revolution. It is the result of a gradual growth, a progressive change, what we term an *evolution*. This is as much a law of history as of nature."

The Ethic of Freethought closes with three lectures on the woman question given in 1885 or 1886. Here he urged that the problem was one which called for study by scientific and impartial minds, and not by platform appeals to "women's rights." There were many questions on which it was futile to give a verdict until patient research had been undertaken. "Is there like or unlike inheritance by male and female children of their parents' intellectual capacity?" Is it not likely "that in the future the best women will be too highly developed to submit to child-bearing; in other words, the continuation of the species will be left to the coarser and less intellectual of its members? This seems to me a very serious difficulty, demanding the most thorough investigation." "Shall the phthisical father not be socially branded when he hands down misery to his offspring, and inefficient citizens to the state?" Before many years have passed we shall find him, with Galton's statistical tools, sharpened, in hand, returning to a serious attack on just these problems.

Though to-day they are to a large extent problems of the past, Pearson's treatment of these social and sex questions was at that time both original and outspoken. For this reason it was immensely stimulating to some, while to others it was abhorrent. Thus we find Olive Schreiner writing of the last three lectures in these terms in *The Pall Mall Gazette*:

> "Written with haste, not as of one who despises his work, but rather of one who seeks to note down breathlessly each point as it flashes on him, fearing it may escape him if he wait an instant; showing a slight inclination at times to treat the whole subject as rather one of statics than dynamics, there runs through these papers an element which we know not where we shall elsewhere find in the treatment of the woman question. It is the intensity of the desire to learn the truth in this matter,...it is this which gives to the work, seemingly dry and pedantic, its deeply human pulsation."

On the other hand *The Glasgow Herald* closed a sarcastic review with this final shot: "Mr Pearson would nationalise land and nationalise capital: he at present stands alone in proposing to nationalise women also"; while *The Spectator* in a three-column criticism made no reference at all to these last three essays. Such is the fate of those who write before their time!

But now we must turn from this description of a personal philosophy to dates and facts. In June 1884, at the age of twenty-seven, Pearson was appointed to the chair of Applied Mathematics and Mechanics at University College, London. He had made at least two unsuccessful applications for professorships before, in 1883 for the chair of Pure Mathematics at the same College, and in 1881 for a chair of Mathematics at Owens College, Manchester. In the event, we may be glad that he had received neither of these appointments. But uncertainty as to the future had sometimes depressed him, and it is not difficult to realise the anxiety that members of his family and some of his friends may have felt during these years of freedom. A letter to Mrs William Pearson has been preserved, written on the night of Karl's election by his friend Robert Parker, who was sharing rooms with him in the Temple. It expresses the situation so admirably that it may be quoted in full.

2 *Harc't Bldgs.*
9 *June* /84.

My dear Mrs Pearson,

I must write you one line of congratulation on Karl's election at University College; next to yourself and him, I think I am the person most glad of his success and in congratulating you I am congratulating myself.

Knowing Karl as I do, I always felt sure that he would some day make his value felt and get something which would *really* suit him, however disheartened his friends might be at momentary failure. And now we can realise too what a great thing it has been for him to have three or four years quite free and occupied in other studies than mathematics; I do not mean that it has at all conduced to his present success, but no doubt it will make him a happier and more useful man and enable him to avoid any taint of that narrowness which one sees so often in, and fears so much for, men who have devoted themselves exclusively to one absorbing pursuit. And besides, great ideas are often suggested outside the range of the special subjects to which they relate, and Karl is returning to science with a fund of such ideas to be worked out and to make him some day as famous as Clifford or any of his predecessors. I have tried to tell you this before, when I have seen you worried because he was worried about the uncertainty of his plans; and now you must never be worried again, for not only have you a son of whom we are all proud, but one who has achieved a position in which he cannot fail to be appreciated by others and to find happiness himself in useful work. I haven't felt so thoroughly glad for ever so long; and am sure I shall never settle to work to-night, though Karl himself is as deep in his books as if nothing had happened.

Ever your sincere friend
Rob^{t} Parker.

1884–1890

The post at University College was concerned very largely in those days with teaching mathematics to engineering students. Pearson, probably in 1880, had spent some time working in the engineering shops at Cambridge, and later, perhaps when deputising for Professor Rowe at University College in 1883, had come into contact with Alexander Kennedy, a man who was afterwards to win a big reputation as a civil engineer. It was Kennedy who, in 1878, had founded at University College the first student laboratory for practical engineering in Great Britain, and the Department of Applied Mathematics and Mechanics worked in close co-operation with the Department of Mechanical Engineering.

Long afterwards Pearson said(1) of Kennedy:

"He and I shared the same lecture theatre and my Cambridge workshop experience led me frequently into his laboratory, the more so as I had taken on the editing of a *History of the Theory of Elasticity and Strength of Materials*. It was Kennedy who really persuaded me to give up Law and return to Mathematics, and finally landed me in Clifford's Chair...at University College."

The duties of the professor included courses on Modern Geometry, Geometrical Drawing and Projection. Besides these special geometrical courses, nearly the whole of Pearson's teaching of Statics, Dynamics and Mechanics was based on geometrical and graphical methods, for he found it possible by their means to exemplify and elucidate physical laws which it would otherwise have been difficult

to render intelligible to minds untrained in complex mathematical investigation. Certain of these methods were taken over from his predecessor, Professor O. Henrici, but as the latter warmly acknowledged, Pearson "extended their use far beyond the limits I had set myself." The experience which he gained in this connection was no doubt invaluable to him in his later popular Gresham College Lectures and it was certainly turned to good account in his teaching of statistics.

Engineering classes even in those days were large, but the young man who could write enthusiastically on history, ethics and socialism was to find that he had no difficulty in maintaining order in a class of forty or fifty students. Of his attitude towards lecturing, he wrote much later to his son:

"I don't think you can say yet about teaching....There is a sense of power and inspiration in holding a class of 80 or 100* highspirited young men, by the interest of your subject and your way of exposing it—when the failure would mean riot—which is not without its reward. One of my proudest reminiscences was a sentence of Sir William Ramsay's in a talk about a colleague's disorders in class: 'You and I, you know, Pearson, are the only men who can hold big classes in complete silence in the College.' I don't know how he knew, but I don't think I ever turned a man out of class...in thirty years. This is not a boast, because it was so practically from the first, and while I am always nervous and in doubt at a public lecture because I have to deal with a new topic and can't foretell my audience, a subject you know well to a daily audience whom you know personally and can watch individually is a different thing. I think one is born to it, and that is why I say it is not a boast, and why you cannot say yet how far you may have inherited it."

An old engineering student, H. Deanesly, has written some impressions of this period:

"I myself attended K.P.'s classes from 1886–89, 50 years ago! They were on (1) Applied Mathematics and (2) Graphical Statics and Graphical Dynamics. For both, his lectures consisted in demonstrations and proofs sketched on the blackboard, while graphical work was done in the Drawing Office. K.P. made each step and stage in the solution of any problem clear and easily followed (there were no text books available); unintelligent memorising held no place in examination work.

"So far as my memory serves, the result was the pleasant and useful conviction that most mathematical problems could be easiest solved and visualised by graphical treatment—a contrast to pure mathematics; advanced pure mathematics was beyond the realistic grip of most engineering students.

"In my opinion K.P.'s methods were secondary to his personality—that was the key to his success, in the keenness and interest his students took in all his classes. Professor Kennedy was a great admirer and friend of K.P.; he used to tell us of K.P.'s work with Todhunter on Elasticity—fancy being coupled with such a man as I.T.!—and then of his taking to the study of Law before becoming Professor of Applied Mathematics at U.C.L. Then of his (Kennedy's) amusement, when they were together on a holiday excursion to Switzerland and Germany, of the way in which K.P. (in German!) got two clergy of opposite camps into combining against himself in argument.

"Little wonder that admiration for such all-round genius made a fine background for such a personality in the capacity of professor.

* The size of the engineering classes had increased considerably in later years.

"So much for my memory; I need hardly tell you that my contact with K.P. for those 3 years has influenced almost daily my work for the last 50 years!"

There can be little doubt that Pearson gave a great deal of time and energy to this College teaching; as a Demonstrator of a later period, Professor M. T. M. Ormsby, writes: "The things that struck me most about K.P.'s teaching were his wonderful clearness as a lecturer, and the extraordinary conscientiousness running through all his work. In all the time I was with him, for instance, he almost invariably put in the *whole* time at every one of the drawing classes, even for first year students, and I do not think he was ever absent for a minute unless it was absolutely unavoidable." Nevertheless the output of original work continued. An early task after assuming the professorship was to prepare for the press the incomplete manuscript of William Kingdon Clifford's *Common Sense of the Exact Sciences* (10) which was issued in 1885. The chapter on *Position* and considerable portions of the chapters on *Quantity* and *Motion* were due to Pearson.

Afterwards much of his spare time during the early years of the appointment must have been occupied with the preparation and completion of the *History of the Theory of Elasticity* (11). Todhunter's unfinished manuscript was placed in Pearson's hands by the Syndics of the Cambridge University Press in 1884, and the first volume (of 924 pages) covering the period 1639–1850 appeared in 1886. As a result of the introduction of physical and technical memoirs and many omitted memoirs of historical interest, about half this volume was due to Pearson. In 1889 he published separately *The Elastical Researches of Barré de Saint-Venant* (14) (278 pages), and completed in 1893 the second volume of the *History*, which was in two parts* (762 and 546 pages respectively). This brought the account of the chief theoretical writers down to date and of all other writers on the subject down to 1860. With the exception of some few articles, he was responsible for the whole of this second volume; the task, involving the examination of several hundred memoirs and the analysis of complicated mathematics, proved without any doubt Pearson's remarkable capacity for hard work and established his reputation as an applied mathematician. A number of other papers on allied subjects were published by him in mathematical journals during the same period.

How much time Pearson continued to give to research on historical and sociological problems after 1884, it is difficult now to say. Seven or eight of the lectures in *The Ethic of Freethought* were delivered between 1885 and 1887, but it is certain that a great deal of the material had been collected before his appointment to the professorship. Thus, for example, the long article entitled "The German Passion-Play: a Study in the Evolution of Western Christianity," published in *The Chances of Death* (7) as late as 1896, was based on notes which he tells us were collected for a course of lectures in 1883. Again *Die Fronica* (5), that collection of the so-called Veronica portraits of Christ which he described as a contribution to the history of engraving in Germany, was published in 1887, but the research had been carried out under Bradshaw's inspiration a few years earlier.

* The de Saint-Venant paper was incorporated in the first part of this volume.

There can, however, be no doubt that in a period when his capacity for hard work was drawn on to the full in College teaching and in research into the theory of elasticity, his emotional energy found an outlet in the planning and delivery of the later lectures of *The Ethic of Freethought* series. With an outlook that was never static, he would see ever from fresh aspects the illustrative value of the material which he had at hand.

He was also at this time an active member of a small club, formed in 1885 "for the free and unreserved discussion of all matters in any way connected with the mutual position and relation of men and women." His essay on "The Woman's Question" was read as the opening paper of the club, and "Socialism and Sex" was a contribution at a later meeting. Reading through the minute book of the club, one is struck by the thorough and serious manner in which this little group of men and women of fifty years ago set about breaking down the conventional barrier which at that date stood in the way of free discussion of the problems of morality and sex. Among the club's guests may be found the names of Annie Besant and of the Russian exiles Tcherkowski and Stepniak. Its fifteen to twenty members included Olive Schreiner, R. J. Parker and T. W. Rhys Davids, and also Maria Sharpe whom Pearson married in 1890 and who was to be his companion for thirty-eight years.

In the upper photograph facing p. 223 we see this young man of thirty-three. He has behind him a firmly developed philosophy of life which will add intense seriousness to his approach to any big problem; he is married, full of self-confidence, with a growing mathematical reputation; he is also a fighter who, if he find a cause after his own heart, will throw himself into it with a tireless enthusiasm.

1890–1900

> Days when the ball of our vision
> Had eagles that flew unabashed to the sun;
> When the grasp on the bow was decision,
> And arrow and hand and eye were one.
>
> George Meredith,
> *Ode to Youth in Memory**.

The appointment of W. F. R. Weldon to the Jodrell Professorship of Zoology at University College, London, in succession to Ray Lankester in 1890, had a profound influence on the course of Pearson's scientific career. From then onwards his activities were gradually directed with ever increasing intensity into a new field. It is true that the inspiration behind this development must be attributed to Francis Galton; as Pearson has said(1):

"After Bradshaw came Francis Galton. In 1889 he published his *Natural Inheritance*. In the Introduction to that book he writes:

'This part of the inquiry may be said to run along a road on a high level, that affords wide views in unexpected directions, and from which easy descents may be made to totally different goals to those we have now to reach.'

* These lines and the others quoted on p. 226 below hung in Pearson's room at College during the last years of his life.

Road on a high level,' 'wide views in unexpected directions,' 'easy descents to totally different goals'—here was a field for an adventurous roamer! I felt like a buccaneer of Drake's days—one of the order of men 'not quite pirates, but with decidedly piratical tendencies,' as the dictionary has it! I interpreted that sentence of Galton to mean that there was a category broader than causation, namely correlation, of which causation was only the limit, and that this new conception of correlation brought psychology, anthropology, medicine and sociology in large parts into the field of mathematical treatment. It was Galton who first freed me from the prejudice that sound mathematics could only be applied to natural phenomena under the category of causation. Here for the first time was a possibility, I will not say a certainty, of reaching knowledge—as valid as physical knowledge was then thought to be—in the field of living forms and above all in the field of human conduct."

Nevertheless in 1890 Pearson was at work on *The History of the Theory of Elasticity* (not to be completed until 1893), and he had still to fashion into that balanced whole, which we find in *The Grammar of Science* (1892), those fundamental concepts of scientific law, of force and matter and life, that had been taking shape in his mind during the previous years. It could hardly be certain in which direction he would have turned when freed from these labours. It was, I feel sure, the personality of Weldon, "teeming with vigour" and eager "to enthuse a mathematician with his project of demonstrating Darwinian evolution by mathematical enquiries," which played a decisive part in drawing Pearson from more orthodox fields of applied mathematics.

Weldon seems not to have taken up his University College appointment until the beginning of 1891; at this time he had already made use of some elementary statistical methods in a paper, "The Variation occurring in certain Decapod Crustacea. I. *Crangon vulgaris**." The paper had been sent to Galton to referee, contact had been established and with the older man's assistance the statistical treatment had been remodelled and some of the conclusions consequently modified. Weldon, realising his own weakness in mathematics, had at once set about increasing his mathematical knowledge by a thorough study of the work of the great French writers on the calculus of probability. It was at this moment that his move to London brought him into touch with Pearson. The two young professors—Weldon was thirty-one and Pearson thirty-four—"both lectured from 1 to 2, and the lunch table, between 12 and 1, was the scene of many a friendly battle, the time when problems were suggested, solutions brought and even worked out on the back of the *menu* or by aid of pellets of bread†." Probably they were battles between two enthusiasts only; as Professor F. W. Oliver, a sharer of the same table, writes: "Tossing pennies and arranging crumbs on the lunch table were always in progress, and formed a regular background to our lunches and pots of beer, but I don't think we were ever initiated into the full significance of these activities."

* *Proc. Roy. Soc.* XLVII. (1890), pp. 445—453.

† This and much of the information regarding those days is taken from Pearson's Memoir of Weldon(2).

In November 1890, before the battles of the pellets had begun, Pearson had already applied for the Gresham Lectureship in Geometry and, as we shall see below, had mentioned statistics and the theory of probability in his outline of a proposed syllabus. His lectures at Gresham College dealing with these subjects did not however commence until the autumn of 1891, and Weldon's infectious enthusiasm certainly influenced considerably the extent to which the subject-matter was expanded.

Another memory of this period is interesting to record. W. H. Macaulay recollects how one day he and Pearson were walking on Hampstead Heath*; they sat on a seat and the latter talked of a subject for his Gresham Lectures. He suggested the doctrine of chances, tossing halfpennies, and Gauss' Law. His friend thought it a very pretty subject, but questioned if there was enough in it, at any rate for elementary lectures; would he not soon come to the end of it? Before we launch out upon the discussion of Pearson's development of that subject and follow him upon a voyage of forty-five years, we must give some further consideration to this Gresham Lectureship and to the writing of *The Grammar of Science* which was closely associated with it.

Gresham College, founded by Sir Thomas Gresham at the close of the sixteenth century, had been organised on the lines of an old mediaeval university with a professor for each of the seven subjects, divinity, astronomy, geometry, music, law, physic and rhetoric. But this old conception of the division of knowledge into separate compartments was dying even at the start of the College, and the first professors of geometry, among whom were included Henry Briggs and Robert Hooke, regarded their subject as the application of mathematical knowledge to all the branches of physical science. In 1890, Gresham College, which had seen within its walls the early meetings of the Royal Society, was but a poor shadow of what its founder had planned, and Pearson in his Probationary Lecture(15) discussed how that old tradition of the application of science to practical life might best be revived in a form suited to the City audience of the day. It was one of his many contributions to the subject of university education. (See also, for example, (16) and (17).)

In applying for the vacant Lectureship in Geometry, he was seeking for another opportunity of conveying to a more or less popular audience the results of his own investigations, the concepts forming at the moment in his active brain. In the past such lectures had, as we have seen, been mainly concerned with mediaeval and renaissance history, with freethought and with sociology; now they were to be of a more definitely scientific and mathematical character. His work at University College had forced him to consider how the elements of Statics and Dynamics could best be presented, freed from the metaphysics and confused thinking of many of the text-books then current, to young students who had no extensive

* Pearson had moved from the Temple to Christchurch Cottage, Hampstead, some little time before his marriage. This small but attractive house stands by the church of that name not far from the Heath. With an increasing family, the Pearsons moved to a larger house near by, No. 7, Well Road, in 1892.

knowledge of mathematics. And now that insistent call which he felt for self-expression drew him towards an even simpler exposition. In his application for the post he wrote as follows*:

"Should I have the honour to be elected Gresham Lecturer, I think this experience [i.e. at University College] would be of service in appealing to the class of audience which is likely to be attracted to Gresham College. I believe that, by legitimate interpretation of the wide meaning of the word *Geometry* as used in Sir Thomas Gresham's time for one of the seven branches of knowledge, courses of lectures on the elements of the exact sciences, on the geometry of motion, on graphical statistics, on the theory of probability and insurance might be given, in addition to purely geometrical courses, which would supply a want felt by clerks and others engaged during the day in the City."

Pearson received the appointment and gave his first course of four lectures in March 1891, followed by a second course of four lectures in April, under the title *The Scope and Concepts of Modern Science*. The subject-matter, developed and enlarged, was reproduced in the first edition of *The Grammar of Science* (1892)†, several of the eight lectures having titles corresponding to the chapter headings of that work.

Rather than attempt myself to give an account of that great summary of fundamental principles, the first reading of which was an event in the intellectual development of many of the older generation of to-day, I have reproduced in an Appendix to this Memoir the syllabuses of seven of these Gresham Lectures‡. The reader who cares to glance through these will perhaps understand something of the scope of the book, and, if my attempt at outlining the earlier development of Pearson's outlook has been successful, will realise that we have here the matured view of the meaning and function of science built up, step by step, from the questionings of "Arthur" twelve years before.

The Grammar of Science was translated into several languages and its influence was felt far outside our own country. Some impressions of a period of twenty years ago have been given me by J. Neyman, who was then studying mathematics under S. Bernstein at the University of Kharkov in the Ukraine:

"We were a group of young men who had lost our belief in orthodox religion, not from any sort of reasoning, but because of the stupidity of our priests. We had simply lost faith in the officially recommended 'Weltanschauung' and were too busy with our mathematics to attempt to construct a new one. We were not freed from dogmatism and were prepared in fact to believe in authority, so far as it was not religious.

"The reading of *The Grammar of Science*, which Bernstein recommended, was striking, because (1) it attacked in an uncompromising manner all sorts of authorities,

* The words are taken from a printed letter of application dated 18 November, 1890, followed by testimonials from W. H. Macaulay, G. W. Prothero, Percival Frost, Edward J. Routh, T. G. Bonney, M. J. M. Hill, O. Henrici, R. D. Roberts, G. Carey Foster, J. E. Nixon, T. W. Rhys Davids, E. Ray Lankester, W. H. Drew and Henry Bradshaw.

† A second and enlarged edition was published in 1900, and the portion dealing with the Physical Sciences, still further enlarged, in 1911. All editions are now out of print.

‡ The syllabus of the eighth lecture is a reproduction of that of the fourth.

and (2) it was the first attempt we had met to construct a 'Weltanschauung,' not on any kind of dogmatic basis, but on reason. At the first reading it was the former aspect that struck us. What could it mean? We had been unused to this tone in any scientific book. Was the work 'de la blague' and the author a 'canaille' on a grand scale, or was he a great man? But our own teacher, Bernstein, had recommended the book; we must read it again. We did so, and came to the conclusion that the second alternative was true.

"Re-read, the single copy in the University passing between many hands, the book had a tremendous influence on us and changed our way of thinking. I cannot recollect the details of the arguments that followed, but I do remember the pleasure we had in catching each other out in dogmatism. One would say: 'You must recognise that...,' and the other reply: 'Why should I? Is it ordered by the police? Don't you think that what you are saying is simply the result of the "routine of thought"?'

"This was in the summer vacation of 1916; in the next session the Revolution was upon us; in our many meetings, with endless discussion on politics, *The Grammar of Science* was frequently mentioned. Whether we joined the 'reds' or the 'whites' or stood by as sceptical onlookers, its teaching remained to influence our outlook."

The next group of Gresham Lectures was entitled *The Geometry of Statistics*, and consisted of three courses, each of four lectures, given in November 1891, January 1892 and May 1892 respectively. After some historical account of the English School of Political Arithmetic, the German School of State-Science and the French School of Probability, the lecturer devoted his time to a very full description of different graphical methods of presenting data. These lectures were plentifully illustrated, the wide range from which data were collected showing the trouble that Pearson must have taken in preparing his work.

The last lecture of the series closed with a pointer towards the courses that were to come next; the final sentences of the syllabus read as follows:

"Example (*c*). Marriage stereogram for Italy. Distribution of husbands of a given age according to the age of their wives. The curve of frequency or probability surface. Influence of chance in the distribution of the ages of husband and wife for large numbers. Transition to the subject of probability, and its relation to statistics."

The four lectures of November 1892, entitled *The Laws of Chance*, are of considerable historical interest. They show how the author of *The Grammar of Science* was led to realise the essential place of the theory of probability among the tools of the scientific investigator; they mark the start of Pearson's career as a teacher of theoretical and applied statistics; and they suggest how that keen mind which always found in the preparation of lectures fresh problems for research was likely to be led further and further into a subject where so much was unexplored.

The first lecture emphasises how statistics and the laws of chance are intimately associated with the foundations of knowledge. The scientific aspect of cause and effect, discussed in the earlier course on *The Scope and Concepts of Modern Science*, was restated. Thus (quoting from the first paragraph of Chapter IV of *The Grammar of Science*) we "see that law in the scientific sense only describes in mental shorthand the sequences of our perceptions. It does not explain *why* those perceptions have a certain order, nor *why* that order repeats itself; the law discovered by

science introduces no element of necessity into the sequence of our sense-impressions; it merely gives a concise statement of *how* changes are taking place. That a certain sequence has occurred and recurred in the past is a matter of experience to which we give expression in the concept *causation*; that it will continue to recur in the future is a matter of belief to which we give expression in the concept *probability*."

Our belief in repetition or routine, which is an essential condition of thought and inference, is in fact really based on statistical experience. A distinction is made between the subjective and objective aspects of chance; "the wider a man's knowledge and experience the more closely his subjective chance approximates to the objective value." Here the question of *a priori* probabilities inevitably comes forward, and the following paragraph from the syllabus of the second lecture is worth quoting:

"In default of all objective statistics, can we make any statement as to the value of a chance? Problem of *a priori* probabilities. Nature and importance of problem. Ignorance of why one event should occur rather than another does not justify our equalising their chances. Need for appeal to experience of some kind. Principle of the 'equal distribution of ignorance.' Objections to it; Bayes, Laplace, Boole, Venn. Reconciliation by Edgeworth of this principle with the axiom that knowledge must be based on some form of experience. Appeal to statistical experience of a rough survey that all values of probability-constants occur, and without special preponderance of any one. Examples. Experiments with the contents of an unknown bag. 'Rough but solid' basis of principle. Practical character of results deduced from it; its sufficiency as a guide to belief. Indication by a special example of the important applications of the principle of equal distribution of ignorance. Is not the assumption that Nature is 'like a bag of balls' tacitly made by those who reject the principle, but yet apply statistics of past success and failure to measure the chance of future events?"

The reader who wishes to obtain a more detailed picture of Pearson's views on these points must turn to Chapter IV of *The Grammar of Science*; the summary I have given does however provide an outline of his approach to the problem of statistical inference. He would use Bayes' Theorem, or a generalisation of it, but it was by an appeal to experience that he would justify the introduction of some *a priori* probability distribution of the unknown values of statistical constants. I do not think however that he felt quite satisfied with this "rough but solid" basis of principle*. Many years afterwards he believed that he had found a satisfactory answer to this problem†, which has for so long been a centre of controversy, but he realised later, I think, that the solution, free from assumption, had still eluded him‡.

* In the summer of 1891 Pearson had been thrashing out this question in correspondence with W. H. Macaulay. Just at this time his first child was born and Macaulay writes playfully: "For the moment there must be a little anxiety in counting whether a new arrival has the proper number of eyes and toes; I suppose this is what has set you on to p's and q's lately. Is it *a priori* equally probable that a child will have any number of toes from one (or zero?) up to infinity? And what are the *a posteriori* probabilities?"

† "The Fundamental Problem of Practical Statistics," *Biometrika*, Vol. XIII. (1920), pp. 1—16, and pp. 300—301.

‡ "Note on Bayes' Theorem," *Biometrika*, Vol. XVI. (1924), pp. 190—193.

After this introductory discussion the lecturer turned to the deduction of the laws of chance and decided that his best course in an elementary exposition was to appeal to a maximum of experimental evidence and a minimum of theory. "Accordingly," he wrote, "we shall use only arithmetic and common-sense." His first data related to some experiments of his own with the tossing of coins, to roulette results from Monte Carlo and to Westergaard's analysis of Danish Lotteries. The audience also were asked to carry out a number of experiments in tossing coins and in guessing lengths, weights, ages, etc.; these results were analysed and discussed in class.

It is interesting to note how the rapid advancement of statistical theory which Pearson was then achieving with Weldon's encouragement was reflected in the development of these popular lectures, both in changing terminology and in the introduction of new methods. For example, in November 1892 variation is measured by the mean error; by November 1893 the *standard* deviation has been introduced and the curve of error becomes the *normal* curve. In this latter course, compound and skew curves are discussed, and by February 1894 "Problems in Evolution" are considered. The final lecture on 11 May, 1894, concludes with reference to the resolution of the human mortality curve into five components, a problem dealt with at length in the first essay of *The Chances of Death*(7).

Pearson did not repeat in a second year his lectures of a previous year*; he was no doubt carried away by his interest in the subject and by the fascination of taking a growing body of students† with him into fresh fields. But the preparation of twelve public lectures a year in this generous style, in addition to his University teaching, made a heavy tax on the time he wished to give to research; it is hardly surprising therefore that he resigned the appointment in 1894.

The Grammar of Science of 1892 contains a single chapter on the subject of Life. This was written before Pearson had begun the analysis of some of Weldon's data, leading to the development of statistical theory referred to in the later Gresham courses. There are, however, two lines of thought which may be traced in this chapter that are of interest. In the first place we are told that "our object in biology is identical with that in physics, namely to describe the widest ranges of phenomena in the briefest possible formulae"; and again, "any biological concept will be scientifically valid if it enables us to briefly summarize without internal contradiction any range of our perceptual experience." Later in the chapter we find presented the outlook of the scientific historian, who believes that a study of the history and development of living forms over long periods will provide "the most wonderful insight into natural selection," when it may be seen how a catalogue of facts "falls into sequences which can be briefly resumed in scientific formulae."

Looking back it is easy to follow where these trends of thought led, almost at once, in action: to an interest in Galton's Law of Ancestral Heredity; to a more

* With the exception (according to the printed syllabus) of the repetition of a lecture on "The Classification of the Sciences," as mentioned in a footnote to p. 213 above.

† The number of his audience appears to have increased from 30 or 40 to about 200.

accurate statement of this Law, involving the development of the theory of multiple correlation; to the testing of its adequacy as a descriptive formula by an extensive collection and analysis of data on inheritance in man and animals; to an approach to the problems of heredity and evolution through a study of long-run mass effects in populations under natural conditions (the actuarial approach), rather than through experimental investigations into the mechanism of inheritance carried out on individuals; to the introduction of precise tools into the study of racial problems, in other words to anthropometry.

The history of the founding and development of the science of biometry during the years 1891 to 1906 has been described by Pearson with so much detail, so much affectionate care tinged with sadness at the loss of his friend, in the Memoir of Weldon(2), and again to some extent in Chapter XIV of *The Life of Francis Galton*(18), that I shall not attempt here another full account of these years. But it is necessary to couple with certain of the more essential facts some appreciation of Pearson's own contribution to that movement, since this aspect of things was given little emphasis in the memoirs which he wrote himself.

It is interesting to note how at this period the development of statistical theory was closely connected with the problem of correlation. Galton's work on regression and correlation described in *Natural Inheritance* (1889) had arisen from his consideration of two different questions, the problems of measuring (1) the influence of heredity, (2) the extent of association between the sizes of two different organs in the same individual. He had realised that the two problems could be treated by the same method, had defined the correlation coefficient, r, and had dealt with the arithmetical procedure needed for its calculation. It is true that he still measured variation in terms of the interquartile distance, but the problem of dealing with the correlation of two variable characters was, in essentials, solved. He was not, however, clear on the problems that arise when more than two correlated characters are considered together, and it was this failure to grasp the implications of the theory of multiple correlation which led at this time to a certain amount of confused thinking in his treatment of the question of ancestral heredity, in spite of all its suggestiveness*.

Weldon, after first approaching the problem of evolution from the view-point of morphology, had realised that a far more profitable line of advance could be made from a study of the variation among individuals and of the differences between local races; and between 1890 and 1893 he had seen the important part that Galton's statistical methods could be made to play in this work. In three papers published in the *Proceedings of the Royal Society*†, he had established: (i) that a number of characters measured on samples from several local races of the common shrimp and the shore crab were distributed approximately in accordance with the law of error; (ii) that the regression between pairs of these characters was linear; (iii) that values of the correlation coefficient for corre-

* See Pearson's account of this matter in (18), Chapter XIV.

† Vols. XLVII. (1890), pp. 445—453; LI. (1892), pp. 2—21; LIV. (1893), pp. 318—329.

sponding organs in different local races were almost equal, and this although the mean and the variation in the characters, measured by the probable error, differed from race to race; (iv) that in the case of one character the distribution was asymmetrical, and that here a method developed by Pearson suggested that the frequency distribution might be split up into two component parts, representing two types or races of individual.

The last of the three papers ended with the following sentences, which as Pearson has already pointed out* were epoch-making in their formulation of the fundamental principles of biometry:

"It cannot be too strongly urged that the problem of animal evolution is essentially a statistical problem: that before we can properly estimate the changes at present going on in a race or species we must know accurately (*a*) the percentage of animals which exhibit a given amount of abnormality with regard to a particular character; (*b*) the degree of abnormality of other organs which accompanies a given abnormality of one; (*c*) the difference between the death rate per cent. in animals of different degrees of abnormality with respect to any organ; (*d*) the abnormality of offspring in terms of the abnormality of parents and *vice versa*. These are all questions of arithmetic; and when we know the numerical answers to these questions for a number of species we shall know the direction and the rate of change in these species at the present day—a knowledge which is the only legitimate basis for speculations as to their past history and future fate."

The methods of analysis at Weldon's disposal at that date were of course crude; the measure of variation used was either obtained from the interquartile distance, found graphically from Galton's ogive, or calculated from the mean error; evidence of goodness of fit was obtained by inspection; in comparing correlation coefficients Weldon could only remark that the differences "are not, as it seems to me, larger than the probable error"; and routine methods of computation had not yet been worked out nor were calculating machines then available.

The extent of Pearson's written contribution to the solution of these problems during the years 1893–1901 is astonishing. The results may be found in the pages of some thirty-five memoirs and shorter papers contributed to the *Proceedings* and the *Philosophical Transactions of the Royal Society*. That capacity for hard work which he had used in research into German folklore and the history of Elasticity and that vital energy which had been expended in the development and discussion of ideas were now directed into this new channel. He had reached a stage when he must work out ideas in practice†.

The main purpose of all this work was the development and application of statistical methods for the study of problems of heredity and evolution; it would certainly be wrong to think of the Pearson of this period as concerned with the development of statistical theory for its own sake. We may trace two important

* (2) p. 19.

† As he once remarked to his daughter and a school friend who were arguing about the place of philosophical and religious discussion in school and university: "At 30 you have your ideas, at 40 you work them out, but at 20, well, you are just mad!"

objectives which he had now in view: the testing of the adequacy of Galton's Law of Ancestral Heredity and the development of theory that would throw light on the influence of selection on the correlation and variability of organs.

Galton's Law, expressed in statistical form, is simply a multiple regression formula giving the expected value of a character, such as stature, in the offspring in terms of the values of the character in his ancestors. Thus it may be written

$$x_0 = a_1 x_1 + a_2 x_2 + a_3 x_3 \ldots,$$

where x_0 is the expected deviation for the offspring from the mean of his generation, x_1 is a linear function of the deviations of the two parents, x_2 a similar expression for the four grandparents and so on. The first problem was to determine how the constants $a_1, a_2, \ldots$ were related to the coefficients of correlation between characters in parent and child, grandparent and grandchild, etc.; this was a problem involving the development of a theory of multiple correlation. The next problem was to ascertain from collected data how far these correlation coefficients assumed fixed values for different characters and different races.

The important place taken by the Ancestral Law in Pearson's mind was undoubtedly connected with his general views on the function of scientific law; on the determination of the *how* rather than the *why*. At the conclusion of a paper addressed to Francis Galton as a New Year's greeting, on 1 January, 1898 (19), we find:

"The above illustration of Galton's Law will suffice to prove the wide extent of its applications. If either that law, or its suggested modification, be substantially correct, they embrace the whole theory of heredity. They bring into one simple statement an immense range of facts, thus fulfilling the fundamental purpose of a great law of nature."

The need for investigation of the other problem I have mentioned, the influence of selection on the correlation and variability of organs, was directly suggested to Pearson by Weldon's Report of 1894 on the death-rate due to selection in the shore crab, made to the Royal Society Committee "for conducting Statistical Inquiries into the Measurable Characteristics of Plants and Animals*."

In the pursuit of these objectives very extensive data bearing on inheritance were collected, and much of the framework upon which the modern theories of mathematical statistics have been built was cast into shape. In this latter connection we must be content with a brief review of a few of the main features of Pearson's work.

The autumn of 1893† saw the completion of the first of the series of papers (20) afterwards entitled "Mathematical Contributions to the Theory of Evolntion." It was an investigation carried out in response to a request from Weldon, concerned with the breaking up of a frequency distribution into two normal components. The paper is particularly noteworthy for its introduction of the method of moments

* *Proc. Roy. Soc.* LVII. (1895), pp. 360—379 and 379—382.

† The year given here (and below) was that in which the paper was presented to the Royal Society; this date rather than that of publication indicates the time at which Pearson was engaged on the work.

as a means of fitting a theoretical curve to observational data. This method is not claimed to be the best, but is advocated from the utilitarian standpoint on the grounds that it appears to give excellent fits and provides algebraic solutions for calculating the constants of the curve which are analytically possible. Definitions of the *normal* curve and of the *standard* deviation are given, and the symbol σ is introduced for the latter. We find included the first of the long series of Pearson's auxiliary tables; namely a table of the powers of natural numbers (1 to 30) up to the sixth, to assist in the calculation of moments, and a short table of ordinates of the normal curve.

In 1894 was completed the paper on "Skew Variation in Homogeneous Material" (21), which introduced Pearson's comprehensive system of frequency curves, linked together by their derivation from a single differential equation. The types of curve were extended in two later supplements (22), (23). The permanent value of this system of curves cannot be seriously questioned; apart from their proved adaptability in graduation problems, their importance in statistical theory has been increased by the later discovery that, provided we are dealing with normal variation, they represent the sampling distributions of so many of the criteria used in statistical tests, e.g. χ^2, s^2, t, r (when $\rho = 0$), the ratio of two variances, etc.

In 1895, in "Regression, Heredity and Panmixia" (24), the work of Bravais, Galton and Edgeworth on correlation is restated and pushed much farther forward in the direction of the solution of problems in multiple correlation. It is interesting to note that a proof is given establishing, by what would now be termed the method of maximum likelihood, that the product-moment expression for r,

$$\Sigma(xy)/\sqrt{\Sigma(x^2)\,\Sigma(y^2)},$$

is the best estimate of the population correlation, ρ.

In 1897 a paper "On the Probable Errors of Frequency Constants and on the Influence of Random Selection on Variation and Correlation" (25) appeared in the joint names of Pearson and one of his mathematical demonstrators, L. N. G. Filon. The theoretical side of this paper was concerned with the derivation of probable errors and intercorrelations for frequency constants calculated from large random samples drawn from a multivariate population. Special results, badly needed in the practical work then in progress, were deduced. The mathematical approach followed very general lines; thus, using a more recent terminology, if

$$p = p(x_1, x_2, \ldots x_m \mid \theta_1, \ldots \theta_c)$$

represents the elementary probability law of m variables $x_1, \ldots x_m$, dependent on c constants or parameters $\theta_1 \ldots \theta_c$, and T_s is an estimate of θ_s calculated from a sample, we find, among other results, the now familiar approximation to the sampling variance of T_s,

$$\sigma^2_{T_s} = -1 \Big/ \int \ldots \int p \frac{\partial^2 (\log p)}{\partial T_s^2}\, dx_1 \ldots dx_m.$$

R. A. Fisher in his work on the Theory of Estimation* has followed a line of advance diverging from this point, and he has questioned why Pearson himself proceeded no further†. Pearson has replied‡ that while he knew in 1897 that if the sample size, N, be increased indefinitely the joint distribution of the estimates of statistical parameters could be represented by a multivariate normal surface, with coefficients expressed in terms of integrals such as that given above, experience had soon shown him that limiting results for the case where $N \to \infty$ were rarely of practical value.

It is, I think, clear that at this period Pearson was far too much interested in pushing forward with the statistical examination of problems of heredity to question very fully the theoretical possibilities of his results. He also does not appear to have thought out fully the implications of the difference between what I have termed T and θ above. His expressions for probable errors were functions of the θ's; nevertheless in turning them to practical use, values of the T's were substituted for the θ's in the formulae, without reference to any assumptions involved§. We are here observing something which is characteristic of the progress of any branch of science; dangers that once understood seem obvious to later workers and openings which might prove fruitful for new research are missed by the pioneer, intent on blazing the trail into undiscovered lands.

Among other theoretical developments, we may notice the introduction of methods of calculating correlation for data classed into broad or qualitative categories (26), (27). The need for such methods arose directly an attempt was made to measure the intensity of fraternal and ancestral correlation for such characters as eye colour, hair colour and coat colour. That there was an assumption in supposing that such characters were a function of some quantitative variable following approximately the normal law of frequency Pearson was fully aware, and his discussion of the suggestive numerical results is full of caution.

The theoretical work of this period was completed by the introduction of the χ^2-Test for Goodness of Fit, one of Pearson's greatest single contributions to statistical theory (28). Looking back now, it may seem an easy step between the elements

$$ce^{-\frac{1}{2}\chi^2}\,dx_1 \ldots dx_n \quad \text{and} \quad c'\chi^{n-1}e^{-\frac{1}{2}\chi^2}\,d\chi,$$

but history has shown how it is often the simple steps which are the most brilliant and far reaching in their consequences. The χ^2-Test provided a single criterion of extreme simplicity with which to measure the aggregate deviation of a number of variates (correlated or uncorrelated), each having its own deviation. On the practical side it led at once to a means of testing on observational data, in terms of a probability measure, the adequacy of a hypothetical law of frequency. On the theoretical side it was the precursor of many similar tests, based on single comprehensive criteria, of complex hypotheses.

* *Phil. Trans.* CCXXII, A (1922), pp. 309—368. † *Loc. cit.* p. 329.

‡ *Biometrika*, XXVIII. (1936), pp. 49—50.

§ At a later date, of course, Pearson returned to this point.

The first four sections of the paper dealt with what may be termed the fundamental χ^2 problem, the case in which the theoretical probability law is known *a priori*. In section (5) consideration was given to the situation which occurs when the theoretical law must be partly determined from the sample itself; this was the problem which some twenty years later was to be the centre of so much controversy.

If the N observations of some character x fall into k groups and (a) n_t is the number falling into the tth group ($t = 1, 2, \ldots k$), (b) m_t the expected number in this group using the true or hypothetical population law, (c) $\tilde{m}_t$ the expected number calculated from the law fitted to the sample, then Pearson pointed out that we must distinguish between

$$\chi^2 = \sum_{t=1}^{k} \frac{(n_t - m_t)^2}{m_t} \quad \text{and} \quad \chi_s^2 = \sum_{t=1}^{k} \frac{(n_t - \tilde{m}_t)^2}{\tilde{m}_t}.$$

Expanding in terms of the ratios $\theta_t = (m_t - \tilde{m}_t)/\tilde{m}_t$, he argued that the leading term in the difference $\chi^2 - \chi_s^2$ was $\Sigma \left\{\theta_t^2 \frac{n_t^2}{\tilde{m}_t}\right\}$, and that as the θ's were of the order of ratios of probable errors to expected values, the difference between these two χ^2's was likely to be negligible. This argument was not supported by any detailed analysis, and Pearson did not, I feel sure, realise that the difference might depend on the number of constants in the theoretical law whose values were determined from fitting to the sample. If, to take the extreme case, there were k of these constants and a suitable method of fitting were employed, every $\tilde{m}_t$ might become equal to n_t; in this case $\chi_s^2 = 0$, and

$$\Sigma \left\{\theta_t^2 \frac{n_t^2}{\tilde{m}_t}\right\} = \Sigma \frac{(m_t - n_t)^2}{n_t} \to \chi^2.$$

I do not know that this underestimate of the magnitude of a difference led to any serious error in the conclusions drawn from subsequent application of the test in practice. But we can see how the volume of the work that Pearson was undertaking at this time led inevitably to a certain hurry in execution; we can realise, too, the value of that advice which in later days he would often press on his students, of supporting theoretical work with the evidence of experimental sampling.

There is another point which may be noted, because of its bearing on Pearson's approach to the general problem of statistical tests. He regarded χ_s^2, as defined above, not as something which in itself might provide a criterion of goodness of fit, but as an approximation to the unknown χ^2 whose value was what he really needed. Thus if at this time he had found that on the average $\chi^2 - \chi_s^2 = c$ approximately, where c is the number of constants calculated from the observations in fitting the theoretical law, it is possible that he would have taken $\chi_s^2 + c$, an estimate of χ^2, as his criterion, rather than refer χ_s^2 to a modified distribution. This is pure speculation, but one point is certain; if we statisticians of to-day differ here from the Pearson of 1900, who was in many things the Pearson of 1920, it is not

1890

1910

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primarily over a mathematical miscalculation, but from a change in our approach to the problem of inference. Such a difference in outlook between a younger and an older generation is of a kind which comes inevitably in the historical development of a subject; we shall touch on the matter again in the later pages of this memoir.

The last section of the paper is devoted to illustrative examples. Two points may be noted: the calculation of χ^2 to very many decimal places (e.g. $\chi^2 = 17{\cdot}775{,}7555$), the typical care-free extravagance of the owner of a Brunsviga, whom we see happily at work in the photograph of 1910 on the opposite page; and a hard knock at the textbook writers on the Theory of Errors. Thus we find in criticism of Airy's "Practical verification of the theoretical law of the frequency of errors" (an Appendix to his *Theory of Errors of Observation*) the following:

"Yet Sir George Airy takes a set of observations, the odds against which being a random variation from the normal distribution are 70 to 1, to prove to us that the normal distribution applies to errors of observation. Nay, further, he cites this very improbable result as an experimental confirmation of the whole theory! 'It is evident,' he writes, 'that the formula represents with all practicable accuracy the observed Frequency of Errors, upon which all the applications of the Theory of Probabilities are founded: and the validity of every investigation in this Treatise is thereby established.'

"Such a passage demonstrates how healthy is the spirit of scepticism in all inquiries concerning the accordance of theory and nature."

χ^2 was indeed a powerful new weapon in the hands of one who sought to battle with the myths of a dogmatic world!

But now we must leave the development of theory, to return to more general matters. Reference has been made above to the extensive collection of data used in Weldon's and Pearson's statistical investigations; widely different characters in very different races were purposely chosen in order to examine how far there was any common law of inheritance. Thus in Man, stature, cephalic index, eye colour, fertility and longevity were considered; in Thoroughbred Horses, coat colour and fecundity; in Basset Hounds, coat colour; in Moths, wing markings; in Daphnia, shape of spine. Throughout all Pearson's work aimed at testing the adequacy of the Ancestral Law, there runs that scientific spirit of inquiry which we might expect from the author of *The Grammar of Science*; testing by experiment, restatement of theory, more trials, fresh difficulties in interpretation and a call for the patient collection of further data. Any reader who wishes to gain some appreciation of the range of problems covered should turn to the two chapters on Evolution added to the second edition of *The Grammar of Science*, published in 1900. That year, in which the work of Mendel was rediscovered, marks the end of a phase in the history of biometry.

Throughout the whole of this period it must be remembered that very much of Pearson's time during the University terms was taken up with his work in the Department of Applied Mathematics. In 1884 he had been single-handed and lectured eleven hours a week; in 1897 he took sixteen out of the Department's

thirty-six hours of lectures (I think the work in the Drawing Office was in addition to this), and was helped by an Assistant Professor and two Demonstrators. Applications which he made for the Savilian Professorship of Geometry and the Sedleian Professorship of Natural Philosophy at Oxford in 1897 and 1899, respectively, show that he was seeking for a change from his London post to surroundings giving greater freedom for research. But it was Weldon, and not Pearson, whom Oxford took in 1899 as Linacre Professor of Comparative Anatomy. Again in 1901 Pearson applied, unsuccessfully, for the chair of Natural Philosophy at Edinburgh. It was not until the financial grant from the Drapers' Company, initiated in 1903, provided a fund which could be allocated wholly to biometric work, that Pearson felt that conditions in London gave him the scope which he needed.

Nevertheless, during the '90's, the presence at University College of mathematical assistants who shared in the teaching of engineering students certainly provided an opportunity for the development of a keen group of statistical workers. Some of the papers presented to the Royal Society were jointly written, and in all of them reference is made to help received, whether in the discussion of theory or in computation. G. U. Yule, who at an early date made his own contributions to the theory of correlation*, was attached to the Department of Applied Mathematics from 1893 to 1899, and was one of those who attended Pearson's first course of lectures on the theory of statistics. Notes of these lectures, taken in the session 1894–95, Yule generously placed some years ago in the Department of Statistics at University College†. "Let us call the occasion," he then wrote, "the thirtieth anniversary of the opening of the first notebook—it is only a few months later. They are I think of interest now, and will be of more historic interest in the future, as showing exactly what Pearson was then teaching—subject always to errors of the scribe."

L. N. G. Filon was also an assistant, and has described (1) how he came to be joint author of the paper on the probable errors of frequency constants (25):

"K.P. lectured to us on the Mathematical Theory of Statistics, and on one occasion wrote down a certain integral as zero, which it should have been on an accepted general principle. Unfortunately I have always been one of those wrong-headed persons who refuse to accept the statements of Professors, unless I can verify them for myself. After much labour, I actually arrived at the value of the integral directly—and it was nothing like zero. I took this result to K.P., and then, if I may say so, the fun began. The battle lasted, I think, about a week, but in the end I succeeded in convincing Professor Pearson. It was typical of K.P. that, the moment he was really convinced, he saw the full consequences of the result, proceeded at once to build up a new theory (which involved scrapping some previously published results) and generously associated me with himself in the resulting paper."

* See for example a paper "On the Theory of Correlation," *Journ. Roy. Stat. Soc.* LX. (1897), Part 4, in which the partial correlation coefficient (then termed the *net* correlation coefficient) is introduced.

† Mr Yule has himself given some description of this period in his memoir of Pearson, just published in the *Obituary Notices of Fellows of the Royal Society*, Vol. II. Part 1 (1936).

Another stalwart worker of these and later days was Alice Lee, for many years a lecturer at Bedford College, who between 1896 and 1927 contributed a wealth of unflagging energy to the work of statistical computation and measurement and to the construction of fundamental tables.

A description of this period would be incomplete without a brief reference to some other events; to the long fight for the reform of the University of London, in which Ray Lankester, Carey Foster, Weldon and Pearson, all of University College, played prominent parts; to Pearson's book published in 1892 entitled *The New University for London. A Guide to its History and a Criticism of its Defects* (16); to his election to the Royal Society in 1896 and to his receipt of the Society's Darwin Medal two years later; finally, to the publication of two volumes of essays, *The Chances of Death and other Studies in Evolution* (7) in 1896.

The contents of these two volumes were as varied as those of *The Ethic of Freethought*, but the common link is to be found in the personality of the author, with his "endeavour to see all phenomena, physical and social, as a connected growth." The essays entitled "The Chances of Death," "Reproductive Selection" and "Variation in Man and Woman" are concerned with the illustration, in simple terms, of the place of the theory of chance and the methods of statistics in the study of social problems. They translate into popular form something of what was appearing elsewhere in the publications of the Royal Society.

In three other essays, that "terrible person K.P." of the pages of *The Academy* and *The Athenaeum* takes up his lance again in the cause of reason against "a new bigotry masquerading in the mantle of science." "Socialism and Natural Selection" is an answer to views expressed by Benjamin Kidd in his *Social Evolution*; in "Politics and Science" Lord Salisbury is taken to task for his Presidential Address to the British Association of 1894; while "Reaction" is devoted to a severe criticism of Arthur Balfour's *Foundations of Belief*. The vigour of Pearson's attack was, as usual, associated with a firm conviction that the ideas with which he was faced were misleading and retrogressive; that the influence which they might have for harm was all the more dangerous because they were seemingly backed by authority; and that therefore it was a duty to spare no effort to bring such teaching to discredit.

In the four essays of the second volume, "Woman as Witch," "Ashiepattle: or Hans seeks his Luck," "Kindred Group-Marriage" and "The German Passion-Play: a Study in the Evolution of Western Christianity," we return to the Pearson of folklore and early German history, who, reading Grimm with interest and delight to his small daughter, turns back again to his work of ten years before; and so, before the urgent call of his science crowds out all time for such distractions, pauses to reshape his notes into a form for publication.

1900–1906

> Do you not know that there are sciences which demand the whole of a man, without leaving the least portion of his spirit free for your distractions?
>
> Michael Angelo.

With the continued collection and analysis of observational data, the outlook of the biometric workers on the problems of inheritance was inevitably expanding and developing. The following quotation from a paper of Weldon's will help to explain their views at the beginning of this period*:

"If two plants, or two animals, of different characters be allowed to breed together, the parental characters may affect the offspring in any one of three ways. In the most usual case, the characters by which the parents differ may appear so intimately blended in the offspring that each young animal or plant appears intermediate in character between its parents, but it is not generally possible for us at present to resolve its body into separate elements, some of which resemble one parent, and some the other. In other cases, however, the body of the young is easily divisible into regions, in some of which the character of one parent is presented in a recognisable and often apparently unaltered state, while the rest of the body presents a similar resemblance to the other parent. In the cases of the third class the body of the offspring may entirely resemble that of one parent, the characters of the other being apparently unrepresented. While it is perfectly possible and indeed probable that the difference between these three forms of inheritance is only one of degree, it is still convenient to discuss them separately.

"The work of Galton† and Pearson‡ has given us an expression for the effects of *blended* inheritance which seems likely to prove generally applicable, although the constants of the equations which express the relation between divergence from the mean in one generation and that in another may require modification in special cases. Our knowledge of *particulate* or mosaic inheritance, and of *alternative* inheritance, is however still rudimentary, and there is so much contradiction between the results obtained by different observers, that the evidence available is difficult to appreciate."

In the case of blended inheritance certain definite results had been established for a number of different characters: (*a*) these characters were approximately normally distributed; (*b*) correlation with linear regression existed between different characters; (*c*) similar correlation also existed between corresponding characters in offspring and parent, grandparent, etc.; (*d*) among living populations mating at random there was evidence of a rough degree of constancy in the values of these correlation coefficients; (*e*) there also appeared to be evidence of a selective death-rate associated in certain cases with these inheritable characters.

Such results had been established by actuarial methods; the simplified description provided in the Law of Ancestral Heredity, while it did not explain the phenomena of inheritance, made it possible to realise what consequences, having a bearing on evolution, might be expected to follow from the mass action of selective forces. It seemed to Pearson and Weldon that the need was to extend this method

* *Biometrika*, I. (1901), p. 228.

† *Proc. Roy. Soc.* LXI. (1897), pp. 401—413.

‡ *Proc. Roy. Soc.* LXII. (1898), pp. 386—412.

of inquiry by the collection of further data, before any attempt was made to interpret the results in terms of the mechanism of inheritance. This attitude is expressed in two quotations from Pearson's Royal Society papers:

"In the first place we must definitely free our minds, in the present state of our knowledge of the mechanism of inheritance and reproduction, of any hope of reaching a mathematical relation expressing the degree of correlation between individual parent and individual offspring. (The physical and arithmetical statements of Weismann's 'Theory of Germ Plasm' offer, so far as I have been able to interpret them, no sound basis for a quantitative theory of heredity in the mathematician's sense.)...We must proceed from inheritance in the mass to inheritance in narrower and narrower classes, rather than attempt to build up general rules on the observation of individual instances." ((24) p. 255, 1896.)

"What I venture to think we require at present is not a hypothetical plasmic mechanics, but careful classifications of inheritance for the several grades of relationship, for a great variety of characters, and for many types of life. This will require not only the formation of records and extensive breeding experiments, but ultimately statistics and most laborious arithmetic. Till we know what class of characters blends, and what class of characters is mutually exclusive we have not within our cognizance the veriest outlines of the phenomena which the inventors of plasmic mechanisms are in such haste to account for." ((27) p. 121, 1900.)

It was precisely at this stage in the progress of inquiry, in 1900, that Mendel's work with its hypothesis regarding the mechanism of individual inheritance was rediscovered. The remarkable simplicity of Mendel's conception at once attracted the attention of many biologists, most notable among whom in this country was William Bateson. Between the Biometric and Mendelian schools there was nothing fundamentally incompatible; in pursuit of the same objective they followed different lines of approach, which were essentially complementary rather than antagonistic. With co-operation a big advance might have been made in the attack on the problems of evolution, and we might have seen many years earlier that combination of genetics, biometry and statistics, the value of which is being recognised more fully to-day.

Looking back after thirty years we can see three main causes which seem to have destroyed any chance of fruitful co-operation: the difference in outlook between the biologist and the mathematician, an incompatibility of temperament among the leaders of the two schools and finally, and perhaps most decisively, the early death of Weldon in 1906. The initial difference in outlook was inevitable; even Weldon, who was essentially a field naturalist with no innate taste for figures or symbols, had not been won lightly to biometry:

"Acceptance of each stage of biometric theory could only be won from Weldon by a tough battle; it had first to justify its necessity, and next to justify its mathematical correctness. He was not drawn into actuarial work by his sympathies or his friendships, he was *driven* into it by the looseness he discerned in much biological reasoning; he felt an *impasse*, which could only be surmounted by the stringency of mathematical logic." ((2) p. 30.)

The introduction of mathematical methods into biology was novel and as such was bound to arouse opposition, calling up that very human characteristic of rationalisation, which classes as unsound and unnecessary that which cannot be readily understood. At the same time the mathematician, no doubt, tended to underestimate the intellectual powers of the biologist, because his processes of reasoning were different. More difficult than this, however, was the personal element.

Early signs of the controversy that was to follow had appeared in the '90's, in connection with the Evolution Committee of the Royal Society. The formation of this committee in 1894* "for the purpose of conducting Statistical Enquiry into the Variability of Organisms," had been largely due to the initiative of Galton and Weldon, and in its early years it was mainly concerned with the biometric researches of Weldon. Much of this work, suggesting that the Darwinian theory might be capable of statistical demonstration, was soon found to be controversial; the difficulty of carrying on such research through a committee became evident, and Galton, as Chairman, struggled with a flood of letters of criticism, many of which were from Bateson. In 1896 Pearson was added to the committee as a statistician; in 1897 nine additional members were elected, including Bateson, who had very little sympathy with the statistical treatment of biological problems, the work for which the committee had been originally appointed. The situation became rapidly impossible, as no agreement on a common line of research could be reached, and in 1900 both Pearson and Weldon resigned, to be followed a year later by Galton. The later work of the committee was devoted to the publication of research on Mendelian lines.

That there should have been much personal bitterness in connection with this incident was inevitable; on both sides emotions must have been stirred up round the committee table which put a great barrier in the way of any further co-operative work. Writing to Galton in 1897 after one of these meetings, Pearson said†:

"I felt sadly out of place in such a gathering of biologists, and little capable of expressing opinions, which would only have hurt their feelings and not have been productive of any real good. I always succeed in creating hostility without getting others to see my views; infelicity of expression is I expect to blame. To you I mean to speak them out, even at the risk of vexing you.

"All the problems laid down by you in your printed paper seem to me capable of solution, and nearly all of them *in one way only*, by statistical methods and calculations of a more or less delicate mathematical kind. The older school of biologists cannot be expected to appreciate these methods, e.g. Ray Lankester, Thiselton-Dyer, etc. A younger generation is only just beginning its training in them."

Another incident of the year 1900 was to increase still further the gap between the biometricians and the more orthodox biologists. In that year Pearson completed a memoir "On the Principle of Homotyposis and its Relation to Heredity, to the Variability of the Individual, and to that of the Race. Part I. Homotyposis in the

* The original members were F. Darwin, F. Galton, A. Macalister, R. Meldola, E. B. Poulton and W. F. R. Weldon.

† See (18), Vol. III. pp. 127—128.

Vegetable Kingdom"(20), which was No. IX of the series *Mathematical Contributions to the Theory of Evolution.* This paper put forward a novel hypothesis on the subject of inheritance, an outline of which can be gathered from the following quotation taken from the introductory section:

"Just as we can find by the methods already discussed in earlier memoirs of this series, the degree of correlation between brothers and the variability of an array of brothers due to the same parentage, so we can determine the correlation, i.e., the degree of resemblance between the undifferentiated like organs of the individual and the degree of variability within the individual. This determination is the answer to our first fundamental problem, that of the ratio of individual to racial variability. But turning to the process of reproduction, the offspring depend upon the parental germs, and it would thus seem that the degree of resemblance between offspring must depend on the variability of the sperm cells and the ova which may each be fairly considered as 'undifferentiated like organs.' Here again we are not compelled to assert that much or little is due to environment and little or much is due to inherent ancestral influence. All we assume is that such causes as produce the likeness between leaves of the same tree, or florets on the same flower, produce the likeness between spermatozoa or ova of the same individual, and that on this likeness the ultimate resemblance of offspring from the same parent depends. We have then to investigate how the quantitative resemblance between offspring of the same parents is related to the quantitative resemblance between the undifferentiated like organs in the individual; and then we must test on as wide a range of data as possible this theoretical relationship.

"Now the reader will perceive at once that if we can throw back the resemblance of offspring of the same parents upon the resemblance between the undifferentiated like organs of the individual, we shall have largely simplified the whole problem of inheritance. Inheritance will not be a peculiar feature of the reproductive cells. One frog, let us say, differs from another in that it produces blood corpuscles more or less alike and unlike those of another frog. In the simplest forms of reproduction, budding and parthenogenesis, the offspring will not be absolutely alike, for buds and ova are undifferentiated like organs, and such organs have only a limited degree of resemblance. If this view be correct, variability is not a peculiarity of sexual reproduction, it is something peculiar to the production of undifferentiated like organs in the individual, and the problems of heredity must largely turn on how the resemblance between such organs is modified, if modified at all, by the conditions of nurture, growth, and environment generally."

It was then shown by mathematical reasoning that if this hypothesis were true, the correlation between brothers should on the average be equal to the correlation between undifferentiated like organs put forth by an individual. The main part of the paper contained a description and analysis of very extensive data collected to investigate this theory. In conclusion 22 homotypic correlations were given, ranging from 0·86 to 0·17 with an average of 0·4570, and these were compared with 19 fraternal correlations collected from previous research investigations, which ranged from 0·69 to 0·20 with an average of 0·4479.

Pearson was careful to use all available correlations in both series, realising that in rejecting doubtful values in order to reduce the range of variation in the coefficients, he might be subject to some bias in the selection. He did not lay any stress on the striking correspondence of the two averages, 0·4570 and 0·4479,

which, having regard to the variation among the coefficients, was somewhat fortuitous, but he did arrive at the following conclusion (*loc. cit.* p. 359):

"What I should accordingly deduce as legitimate from the above general results would be this, the intensity of pure homotyposis throughout the vegetable kingdom probably lies between ·4 and ·5; this is also the mean value found up to the present for fraternal correlation. We may accordingly conclude that heredity is really only a phase of the wider factor of homotyposis. At bottom it is only part of the principle that when an individual puts forth undifferentiated like organs these are not exactly the same, but with a definite intensity of variation have a definite degree of likeness. When we associate heredity with sexual reproduction, we are only considering the result of homotyposis (variation and likeness) between individual spermatozoa and between individual ova. Such homotyposis leads to a likeness of the individuals resulting from the zygotes, which has probably the same mean value as homotypic correlation itself. Thus if the continuity of the germ cells between parent and offspring be realised, we face no longer the problem of heredity, but that of homotyposis, and this again might possibly be reduced to the simplest problem of budding or cell multiplication. Why does the fundamental life-unit on self-multiplication produce homotypes with a definite degree of likeness and a definite degree of variation? I shall hope for further light on this problem when my data for homotyposis in the animal kingdom, already being collected, are somewhat more complete; but only the biologist, not the mathematician, can solve it."

The collection and reduction of the material used in this paper were on a larger scale than Pearson had previously attempted, and they had been only possible through the willing co-operation of many friends and colleagues during the vacations of 1899 and 1900. Among the observations made were: counts of the veins on 26 leaves from each of 100 beech trees; similar counts for leaves of the Spanish chestnut; counts of the stigmatic bands on the seed capsules of Shirley poppies (in one series there were 4443 capsules spread over 176 plants, leading to what has since been termed an "intra-class" correlation table of 197,478 entries); counts of seeds in the pods of broom (e.g. 10 pods from each of 120 bushes); counts of veins in the successive tunics of onions (there were 1085 tunics belonging to 200 onions; a trying job which fell to the lot of Dr Alice Lee).

The result was a memoir of ninety-five pages containing forty-nine tables. The conception and planning were Pearson's, though Weldon had helped with criticism, suggestion and encouragement. "You have got hold," he wrote in the autumn of 1899 ((2) p. 34), "of the big problem which all poor biologers have been trying for ever so long. I wish you luck with it." If the work bore traces of the hand of a mathematician poaching in the fields of biology, and ended with a rather too confident assertion of discovery, a big idea was nevertheless involved*, and much data provided which should have given food for thought to the biologist who could appreciate the significance of the statistical approach. But the biologist and the biometrician were in those days speaking in different terms, or in some cases indeed using the same terms with different meanings, and it was not therefore

* Bateson, in his reply referred to below, before turning to detailed criticism remarked: "It is impossible to write of Professor Pearson's paper without expressing a sense of the extraordinary effort which has gone to its production and of the ingenuity it displays."

surprising that Bateson, as one of the Royal Society's referees, should return a critical report on the paper. Nor need we wonder if the Council and Officers of the Society were somewhat troubled at the length of a contribution which appeared likely to arouse heated controversy.

That there should be free and full discussion on the subject, "stalwart blows" given and received, was to the advantage of scientific progress, but unfortunately, for reasons which we need not now attempt to disentangle, an element of apparent unfairness on the part of authority was introduced into the situation. Pearson's paper, "received" on 6 October, was "read" on 15 November, 1900, when I presume only the abstract was available; the whole memoir was not published until a year later, in November 1901. In the meantime, with the approval of the Zoological Committee, Bateson's detailed criticisms had been printed and issued to the Fellows at a meeting before the criticised paper itself was in their hands and before its author had been notified whether it would be published or not*.

I have mentioned this incident, partly because it shows how heavily the scales were weighted from the start against co-operation, and partly because I think it throws light on Pearson's later attitude in other matters. After considerable correspondence leading finally to an appeal to the President of the Society, on a question of general principle rather than individual unfairness, Pearson gained his point. But the victory was won at some cost. Just as twenty-four years earlier the contest with authority at King's on the question of compulsory divinity had led to depression in the victor, so now I think that this incident tended to give Pearson a feeling that authority was against him and to make him suspect hostile motives, where none may have been present. Because he knew that he would often feel called upon to protest, knew that he could not help striking hard and yet had no liking for the emotion which his own intensity of purpose aroused both in himself and in his opponents, nor for the waste of energy involved, he became less and less inclined to take part in the discussions and activities of scientific societies. It was, probably, largely for these reasons that he never became a member of the Royal Statistical Society and that after 1904 he attended only one further meeting of the British Association†.

As an immediate result of the discussion of Pearson's Homotyposis paper at the Royal Society on 15 November, 1900, came the formulation of plans for the founding of *Biometrika*. Weldon's letter to Pearson, written next day from Oxford, contained the following passage:

"The contention 'that numbers mean nothing and do not exist in Nature' is a serious thing which will have to be fought. Most other people have got beyond it, but most biologists have not....

"Do you think it would be too hopelessly expensive to start a journal of some kind?"

* Bateson's paper entitled "Heredity, Differentiation and other Conceptions of Biology: a Consideration of Professor Karl Pearson's paper 'On the Principle of Homotyposis,'" was "read" on 14 February, 1901, and published in *Proc. Roy. Soc.* LXIX. (1901), pp. 193—205.

† Bateson was President of the Zoological Section in 1904 and his Address, followed by a heated discussion, was directed mainly against the biometric school and its work.

An account of the early history of this Journal will be found in the Weldon Memoir (2), and also in the *Life of Francis Galton* (18) (Vol. III, pp. 100—102); there is much too which can be read between the lines of the first Part of Volume I of *Biometrika*. The value of an independent means of publication to a group of enthusiasts working along original lines was great; it made it easier for them to present their work as a whole and to show the common purpose that ran through it. Also it removed that source of irritation which is bound to arise when the papers of first class minds must be judged by referees whom the writers do not consider fully competent to be their critics.

The approach of the biometricians to Mendel's theory was critical, but not I think biased. Their own extensive investigations had led to a number of conclusions and in this process Galton's Ancestral Law, when put to the test of numbers, had itself been found to need considerable revision. It was natural therefore that they should question critically how far on numerical evidence this new hypothesis appeared adequate to explain observed phenomena. The reader who cares to study Weldon's first review of the subject, entitled "Mendel's Laws of Alternative Inheritance in Peas*," and afterwards Bateson's reply†, will realise I think the extent to which a new element was introduced into the controversy by the latter. It is likely that Bateson was not aware of the painful impression which his style of writing produced; indeed at the beginning of 1902, just before his attack on Weldon was published, he made an effort to bridge a widening gap between himself and Pearson in a long letter written to the latter‡. Forces that were too strong, however, for control drew those vigorous personalities into conflict, and the next four or five years witnessed a controversy which, while it played its part in sifting the chaff from the grain, wasted on both sides much time and energy.

Pearson's further contributions to evolutionary theory during this period were extensive; a few papers only can be selected for mention. In 1901 he completed a further paper of the Royal Society series (30) dealing with the influence of Natural Selection on the variability and correlation of organs. This involved a theoretical study of the effect of the selection of one or more characters on the variation and correlation among a larger number of normally correlated characters. It was illustrated by some data giving measurements on the long-bones of Man, and of the long series of *Mathematical Contributions to the Theory of Evolution* published in the *Philosophical Transactions*, it was the last that contained extensive biological data§.

* *Biometrika*, I. (1901), pp. 228—254.

† *Mendel's Principles of Heredity: A Defence.* Cambridge University Press (1902).

‡ This letter, written from Cambridge on 13 February, 1902, is of considerable biographical interest, but I think too full of references to points then at issue to quote in isolation.

§ The Royal Society Committee of Papers seem to have been troubled for some time by the character of Pearson's papers, which they were unable to class either as mathematical or biological. It was in connection with this paper that in June 1902 they passed the following resolution: "The Committee consider that it is desirable that the Author should make a definite separation between the parts of his paper which are original contributions to mathematics and those which are occupied with illustrations drawn from problems of evolution; and that he should reconsider the amount of the latter illustrations, as it

In 1903 a further paper was written (31) on the Law of Ancestral Heredity. The object of this paper was to clear away certain misconceptions by explaining the statistical character of the Law and providing illustrations of the extent to which it described observed experience in cases which did not appear explicable under the Mendelian hypothesis as developed at that date. The following are some points emphasised in the concluding section:

(*a*) "In all cases as those of man, horse and dog where parents of identical character do not produce identical offspring, the theory of statistics shows us that closer prediction may be obtained when we predict from many instead of few relatives...." [In other words, we gain information by a knowledge of ancestry beyond the parents.]

(*b*) "The law of ancestral heredity in its most general form is not a biological hypothesis at all, it is simply the statement of a fundamental theorem in the statistical theory of multiple correlation applied to a particular type of statistics. If statistics of heredity are themselves sound the results deduced from this theorem will remain true whatever biological theory of heredity be propounded."

A natural step from this paper and one which demonstrates perhaps more clearly than anything else how open were the minds of Pearson and Weldon, was that which led to the Royal Society paper completed in 1903, with the title "On a Generalised Theory of Alternative Inheritance, with Special Reference to Mendel's Laws" (32). The authors'* purpose was to show that there was no essential antagonism between the Biometric and Mendelian methods of approach, and the paper developed what would now be termed a multiple factor theory, in which it is supposed that a somatic character in the individual depends on additive contributions due to a number of independent segregating elements. Starting from this generalised Mendelian hypothesis it was proved that the coefficients of ancestral correlation in populations mating at random would obey certain laws; further that there would follow a system of inheritance which, studied statistically from the actuarial point of view, would in its broad features be essentially the same as that which had "been biometrically developed, not from theoretical hypotheses, but from the statistical description of observed facts in populations."

The hypothesis of the "pure gamete" propounded did not lead to actual numerical values of the coefficients of heredity which accorded with observation. This no doubt was due to the fact that the initial hypothesis, while more general than that of the Mendelian workers of that date, was itself inadequate. But the investigation led to a conclusion of fundamental importance, the moral of which has been overlooked again and again by critics of the biometric methods and by other writers on Mendelian theory†:

has been suggested that the amount of space devoted to them and the detail in which they are treated are more suitable to a separate treatise than to a memoir." The founding of *Biometrika* was amply justified!

* The paper seems to have been a joint work, but Weldon persistently refused to allow his name to appear because he had taken no part in the more complicated algebraic analysis.

† The origin of much misunderstanding has been a different use of terms. Thus we find the following statement made by a well-known biologist, writing on Eugenics in 1927: "Certainly modern genetics gives no support to the view that the somatic characters of the offspring can be predicted

"We thus reach the point we have so often had to insist upon: that the biometric or statistical theory of heredity does not involve a denial of any physiological theory of heredity, but it serves in itself to confirm or refute such a theory. Mendelian formulae analytically developed for randomly mating populations are either consistent or not with the biometric observations on such populations. If they are consistent, it shows their possibility, but does not prove their necessity. If they are not, it shows they are inadequate. The present investigation shows that in the theory of the pure gamete there is nothing in essential opposition to the broad features of linear regression, skew distribution, the geometric law of ancestral correlation etc. of the biometric description of inheritance in populations. But it does show that the generalised theory here dealt with is not elastic enough to account for the numerical values of the constants of heredity hitherto observed." ((32) pp. 85—86.)

During these years a number of breeding experiments were undertaken and further biological data collected. The Weldon Memoir (2) and the early pages of *Biometrika* contain many references to work on snails, moths, mice, Shirley poppies, *Draba verna*, etc. From evidence thus provided and from a critical survey of the results of genetic experiments reported by other workers, Weldon, with Pearson's mathematical aid, was attempting to develop a determinantal theory of inheritance which would provide that increased elasticity which they realised was necessary in any adequate model. Such a theory was to include "simple Mendelism at one end of the range and blended inheritance at the other." The work was not completed at Weldon's death, but the stage which it had reached is indicated in Pearson's article of 1908 written from rough notes that Weldon had left (33). The relation of this hypothesis to the genetic theory of to-day would provide an interesting study.

In tracing the development of Pearson's and Weldon's ideas concerning the theory of evolution it is possible to draw a parallel from the physical sciences. The introduction of a descriptive model of the individual atom with its revolving electrons has been of immense value in the interpretation of the mass phenomena of practical physics; yet research into the behaviour of atoms in the mass—the statistical method of approach—had led to results of far-reaching significance in practical life long before any adequate model of the atom was available. So in the case of the theory of evolution the biometricians realised that the statistical line of attack, involving a study of populations under natural conditions, would lead to results of practical value which need not wait upon the construction and exhaustive testing of any detailed model giving an ordered picture of the process of individual inheritance.

If I have laid special emphasis on the controversies surrounding the development of theories of heredity and evolution, it is because I feel certain that it was the element of battle, as well as enthusiasm in the search for knowledge, which

from a knowledge of the somatic characters of the parents." The writer undoubtedly meant that exact prediction in the case of an individual is not possible. Yet in terms of the statistical concept of correlation prediction is possible. Indeed genetic theory, if adequate, must account for the existence of correlation between the somatic characters of the offspring and of the parent and, when mathematically developed, must lead to the observed numerical value of such correlation coefficients.

braced both Weldon and Pearson at this time to great intellectual activity and to the planning of ever widening programmes of investigation. The holiday workers who collected in Oxfordshire or Yorkshire, when the routine of University terms was over, to search for snails, to count stigmatic bands on Shirley poppies, to grind Brunsvigas and to discuss the week's work at a "Friday biometric tea," were co-operating with enthusiasm in the search for new things. But there runs also through the written accounts and the memories of these years a spirit of comradeship in battle, whether against the assaults of human foes or the perverseness of Nature, unwilling to disclose her Laws.

Through these times of stress both Pearson and Weldon gained much from the help and sympathy of Galton, an honoured figure of an older generation, who stood to some extent outside the field of controversy. It was from Galton too, that "wise old man"—and one might say that young old man—who at eighty years could still launch his Eugenic Creed with much of the vigour of youth, that Pearson drew the inspiration which was to send him into fresh fields of activity.

Research into the inheritance of physical characters in Man had been very naturally followed by research into the inheritance of mental and psychological characters. Galton, in his *Hereditary Genius* and *English Men of Science* had pointed to the existence of a marked inheritance of mental characters, but as Pearson wrote at the beginning of his Huxley Lecture of 1903*, "we require to go a stage further and ask for an exact quantitative measure of the inheritance of such characters and a comparison of such measure with its value for the physical characters." It was clear that the measurement and comparison of mental characters was a far more difficult problem than that of the physical. At the beginning of the Huxley Lecture (35) Pearson gives his reasons for deciding that the best method of attack was to obtain teachers' estimates of the mental and moral characters of school children, as well as physical measurements on the same children, and then to study the intensity of inheritance indirectly through *fraternal* resemblance. "My argument," he wrote, "was of this kind. Regarding one species only, then if fraternal resemblance for the moral and mental characters be less than, equal to, or greater than fraternal resemblance for the physical characters, we may surely argue that parental inheritance for the former set of characters is less than, equal to, or greater than that for the latter set of characters."

Schedules were issued to teachers with careful instructions as to how the measurements and estimates were to be made, and between 5000 and 6000 returns were received. Owing to pressure of other work the collection and reduction of the data were spread over some six or seven years. The physical characters considered were: health, eye colour, hair colour, hair curliness, cephalic index, head length, head breadth, head height and athletic power. The mental characters were: vivacity, assertiveness, introspection, popularity, conscientiousness, temper, ability and handwriting. It was necessary in the latter case, where the data could only be

* "On the Inheritance of the Mental and Moral Characters in Man, and its Comparison with the Inheritance of the Physical Characters," issued first as (34) and reprinted in *Biometrika* (35).

classed in broad qualitative categories, to use the method of the fourfold table which Pearson had developed for calculating correlation for characters not quantitatively measurable* (26). In the *Biometrika* paper ((35) pp. 154—155), two tables show in comparison the fraternal correlations for nine physical and eight mental characters; in each case there are three coefficients, brother-brother, sister-sister, and brother-sister. All values, with the exception of those for athletic power, which Pearson gave reasons for believing must be treated separately, clustered fairly closely round 0·5. Here are the average coefficients (omitting the coefficients for athletic power):

Physical characters:

Brother-brother 0·54; Sister-sister 0·53; Brother-sister 0·51.

Mental characters:

Brother-brother 0·52; Sister-sister 0·51; Brother-sister 0·52.

Pearson's conclusions, drawn from considered reflection on these results, are important because they form an essential step in the argument which runs through so much of his later work on the relative influence of heredity and environment:

"It has been suggested", he said ((35) p. 156), "that this resemblance in the psychical characters is compounded of two factors, inheritance on the one hand and training or environment on the other. If so, you must admit that inheritance and environment make up the resemblance in the physical characters. Now these two sorts of resemblance being of the same intensity, either the environmental influence is the same in both cases, or it is not. If it is the same, we are forced to the conclusion that it is insensible, for it cannot influence eye colour. If it is not the same, then it would be a most marvellous thing, that with varying degrees of inheritance, some mysterious force always modifies the extent of home influence, until the resemblance of brothers or sisters is brought sensibly up to the same intensity! Occam's razor will enable us at once to cut off such a theory. We are forced, I think literally forced, to the general conclusion that the physical and psychical characters in man are inherited within broad lines in the same manner, and with the same intensity. The average home environment, the average parental influence, is in itself part of the heritage of the stock and not an extraneous and additional factor emphasising the resemblance between children from the same home.

"But we are not yet at the end of our conclusions. By assuming our normal distribution for the psychical characters we have found, not only self-consistent results—linear regression, for example, as in the case of the inheritance of intelligence, but we have found the *same* degree of resemblance between physical and psychical characters. That *sameness* surely involves something additional. *It involves a like heritage from parents.* The degree of resemblance between children and parents for the physical characters in man may be applied to the degree of resemblance between children and parents for psychical characters. We inherit our parents' tempers, our parents' conscientiousness, shyness and ability, even as we inherit their stature, forearm and span."

* The mental characters were in fact classified in many cases into more than two categories, e.g. for temper, there was "quick", "good-natured", and "sullen", and for ability there were six groups. Several alternative divisions were tried in making the fourfold tables and close agreement in resulting correlations gave weight to the values adopted.

Mental and physical characters being inherited, the next step in the argument for the development of an active eugenic faith was clear, but only touched on lightly in the Huxley Lecture (*loc. cit.* p. 159):

"The mentally better stock in the nation is not reproducing itself at the same rate as it did of old; the less able, and the less energetic, are more fertile than the better stocks. No scheme of wider or more thorough education will bring up in the scale of intelligence hereditary weakness to the level of hereditary strength. The only remedy, if one be possible at all, is to alter the relative fertility of the good and the bad stocks in the community."

More substantial evidence on this point was provided a year or two later by the work undertaken by David Heron under Pearson's direction, and published from the Biometric Laboratory as the first of the "Studies in National Deterioration*."

The argument based on a comparison of mental and physical characters needed perhaps some reinforcement. The correspondence in the two series of correlation coefficients was remarkable. Yet it was still, I think, possible that the agreement was to some extent fortuitous and that the values of the correlations for mental characters did depend in part on environmental influence and in part upon an hereditary influence, less strong than for the physical characters. A critical test would need to be applied to brothers and sisters separated at birth into different environmental conditions. Such inquiry on statistical lines has I believe been attempted recently in America. It should not be thought, however, that Pearson based his views on the intensity of heredity of mental characters only on the evidence of this school inquiry; in the next few years the problem was to be further examined on fresh series of data.

In connection with Pearson's approach to the problem of National Eugenics, it is of interest to read the small pamphlet entitled *National Life from the Standpoint of Science* published in 1905 (38). This contains a public lecture which he gave in Newcastle on 19 November, 1900, and three Appendices dealing with later correspondence, lectures and research. The main lecture is interesting because we can trace in it a considerable change from the outlook presented in *The Ethic of Freethought* and in some of the lectures of *The Chances of Death*. The writer has matured, has passed from forming ideas to working them out in practice and presents to his audience the real rather than the ideal. He is no longer concerned with the faith of the scientist or sociologist, but with the facts, and sometimes unpleasant facts, of experience which it is his duty as a scientist to set before the nation as a warning.

There is a strain of pessimism, now, in his discussion of national and social problems, which was undoubtedly associated with emotions roused by the South African War. That realisation of failure, of national shortcomings and muddle-headedness which was brought home to the country in 1900 was certainly shared

* "On the relation of fertility in man to social status, and on the changes in this relation that have taken place during the last fifty years." *Drapers' Company Research Memoirs*, Studies in National Deterioration, I (1906).

fully by Pearson. There was aroused in him a national self-consciousness, a feeling of patriotism, with an outlook on empire and the use of colonies which we should perhaps not expect from a study of his earlier writings. There was less of the joyous faith in man's destiny which runs through *The Ethic of Freethought*. The lesson which he now read from history, reinforced by his scientific appreciation of the forces of inheritance and natural selection, was that in one way only had a state of high civilisation been produced, namely by the struggle of race with race, and the survival of that which was physically and mentally the fitter. Such a struggle he regarded as inevitable, and it became therefore vitally important to Britain as a nation to take stock of her physical and mental equipment, and to recognise the need for conscious action directed towards improving the parentage of the citizens of the future. If this course were not followed, the prospects seemed dark.

In later years, after the strong emotions caused by the War had receded, I think some trace of this pessimism remained. He had no longer his youthful belief in social progress towards the ideal, he tended to see in the struggling efforts of governments the things that were missed rather than those that were gained, and in his own field of applied science he was less hopeful than Galton of any early triumph of the eugenic creed. Surely he was in this mood when he fixed for the title of one of the long series of research memoirs, "Studies in National Deterioration."

An event of considerable importance that occurred during this period was the allocation to Pearson's Department of a grant of £1000 by the Worshipful Company of Drapers, one of the old chartered companies of the City of London. This grant was made in the summer of 1903, and was allocated to cover a two-year period. In 1905 a further grant of £2000 was made, this time for a five-year period, and these grants, afterwards fixed at £500 per annum, continued up to the session 1931–32.

The value of this financial contribution was considerable. The money was not spent on any single objective, but gave an increased vigour and activity to every part of the many-sided work which was being carried on at University College under Pearson's direction. I have included as Appendix II to this memoir the Report which Pearson wrote to the Principal of University College after one year's experience of the Grant. This Report not only provides an excellent picture of the activities of the Department at that time, but it shows the extent to which an annual benefaction of this kind, when well spent, can oil the wheels of scientific research. It was a sad day for Pearson when, twenty-nine years later, the Drapers' Company felt compelled to cease their contribution, but he was ever mindful of the great help they had given him in making possible the founding and continuance for so long of his Biometric Laboratory.

It must be remembered that up till 1911 Pearson and his assistants were still teaching Applied Mathematics for the Science and Arts degrees of the University, and were also in charge of the work in Graphics for the Engineering degrees.

Pearson had also launched out on the teaching of Astronomy and through his initiative money was obtained for the building of two small observatories for student training on the front lawns of the College. The first was a Transit House, and later an Equatorial House was added.

Some of his own work in Applied Mathematics and that of his assistants and post-graduate students appeared in the Technical Series of *Drapers' Company Research Memoirs**. One of these memoirs, "On some Disregarded Points in the Stability of Masonry Dams," of 1904 (37), followed by a second of 1907 (38), caused a good deal of interest and controversy at the time. As Weldon wrote from Italy ((2) p. 47), "Your work on dams has filled the Italian papers with horror. They say you threaten the safety of all existing dams, *however long they have stood!*" The final opinion of engineers was, I think, that while the mathematical work was excellent, the experimental work, being based on models made of gelatine, was not altogether applicable to their problems. A friend remembers Pearson referring afterwards to the controversy in a half-humorous vein, telling him how the Government, not wishing at the time to build the Assuan Dam (probably from shortage of funds), took advantage of his paper to postpone it.

Having dealt at some length with Pearson's application of statistical method during this period, I must conclude with a brief survey of some of his contributions to statistical theory, which were almost invariably called forth by the need for solution of some practical problem.

In 1901 (22) he completed a paper which extended his system of skew frequency curves; the main paper had been published in 1895 (21). In 1902 he wrote a paper, which appeared in two parts, (39), "On the Systematic Fitting of Curves to Observations and Measurements." The method advocated was the Method of Moments, and writing with eight years' practical experience behind him he showed, with examples, how it provides a practical solution where other methods fail. He emphasised how much depends both on the way in which the observational moments are calculated and corrected and on the selection of the appropriate type of curve for fitting.

The first introduction of the correlation ratio, η, appears in a paper of 1903 (40). A full discussion of the properties of this measure of correlation was not published until two years later, but the ratio seems to have been calculated in Pearson's Laboratory for some years and compared with the coefficient of correlation, r.

A short paper of 1903 (41) gives expressions for the standard deviations† of certain frequency constants and their intercorrelations; some of these results are exact and some are limiting values for large samples. The multiplication of the standard deviation by the factor 0·6745 to obtain the probable error, while deprecated, was accepted as a long-adopted convention. We may notice again (see p. 221 above) that no reference is yet made to the fact that while the

* For an account of the origin and objectives of these series of Memoirs, see section (iv) of Appendix II, p. 256 below.

† The term *standard error* was not then used.

expressions for standard errors depend on the population parameters, the practical worker must in general substitute sample values into these expressions. That danger might arise from this cause was probably not realised as long as the methods developed were aimed at the analysis of mass statistics.

In 1904 we find an elementary proof of Sheppard's formula for the correction of raw moments (42), probably derived from lecture notes. The same year saw the publication of the first of the Biometric Series of *Drapers' Company Research Memoirs*. This paper "On the Theory of Contingency and its Relation to Association and Normal Correlation" (43) was directed towards solving the problem of obtaining a measure of association between two variables, in cases where the characters are not necessarily classified on a quantitative scale. Thus if ϕ^2 denote the mean square contingency in a sample of N observations, it is shown:

(*a*) how $N\phi^2 = \chi^2$ may be used to test the independence of two characters*,

(*b*) how, upon certain assumptions

$$C_1 = \sqrt{\phi^2/(1+\phi^2)}$$

tends to the value of the correlation coefficient r.

With regard to (*b*), no correction for the number of groups had yet been developed, but it was realised that a valuable alternative to the four-fold method of estimating r was available.

The second paper in the Biometric Series was published in 1905, and was entitled "On the General Theory of Skew Correlation and Non-linear Regression" (44). This paper

(*a*) deals with the properties of the correlation ratio, η, and shows for the first time the fundamental importance of the expressions $(1-\eta^2)\sigma_y^2$ and $(\eta^2 - r^2)\sigma_y^2$, and of the difference between η and r in measuring departure from linearity of regression;

(*b*) attempts to find the probable error of η, but reaches an expression which later work has shown to be somewhat inaccurate;

(*c*) classifies different types of array variability and different forms of regression lines that may arise when the relation between two variables cannot be represented by the bivariate normal law;

(*d*) obtains formulae for parabolic, cubic and quartic regression lines, giving expressions for their constants in terms of the moments and product moments of the observations;

(*e*) shows what conditions must be satisfied for (i) linear, (ii) parabolic and (iii) cubic regression equations to be adequate.

These two papers contain, I think, Pearson's most important contributions to

* If P is obtained from Elderton's Tables, which had then been recently published in *Biometrika*, I. (p. 155), entered with n' = number of cells in the contingency table, it is stated that $1-P$ is a measure of the deviation from independent probability, but no question of the sampling distribution of this form of χ^2 was considered.

statistical theory up to that date since the χ^2 paper of 1900. The first paper introduced the very fundamental conception of contingency, with its extension of the χ^2 criterion to manifold classifications. The second provided novel methods of describing the relationship between two variable quantities, not normally related, and opened the way to the use of many fresh criteria suitable to test the adequacy of this description. If to-day statistical theory has advanced a further step by providing the sampling distributions of these criteria, albeit often under simplifying assumptions, it was the first step made thirty years ago which was the most difficult and most original.

Such must be the record of the years 1900–1906; it is far from being a full record, for Pearson contributed in this period to some seventy published papers. I have not, for example, touched on the work on anthropometry, to which I shall return, nor referred to the work on meteorology. Added to this was most of the routine work of the editorship of *Biometrika*. But if that remarkable constitution, which belied the fears of weakness in youth, could stand the strain, the pace was to prove too great for Pearson's friend and colleague. On 13 April, 1906, Weldon died of pneumonia at a nursing home in London after a few days' illness. His early death at the age of forty-six was a heavy blow to science, and to Pearson one of the greatest personal losses of his life.

From the scientific standpoint it is likely that the event had a profound influence on the immediate course of development of evolutionary theory. The fight for recognition of the place of mathematics and statistics in the study of inheritance and evolution was a hard one. In 1906 it was not won, but Weldon at Oxford was gradually gathering round him a group of young biologists who were infected by his enthusiasm and sympathetic to his views. The advance of biometric methods in evolutionary work needed however the co-operation of the first-class biologist and the first-class statistician. After Weldon's death it was perhaps inevitable that Pearson, feeling that his partner could not be replaced, should turn more and more to other applications of the methods they had developed together; these he found in the field of Eugenics.

It is also, I think, true that while Weldon, the biologist, would have picked out what was of permanent value in the geneticists' attempt to provide an orderly description of how inheritance takes place, Pearson, the statistician, had neither the time nor perhaps the training to attempt this task alone. In the growing complexity of the Mendelian hypothesis, demanding more than ever in his view a stringent statistical examination of the inferences that were logically justifiable, he could not see those simple descriptive formulae which held so important a place in his conception of scientific law. And so he stood aloof, sceptical and often critical.

In the years that followed it was easy for the older biologists, freed from a critic among their ranks, to regard the biometricians as mathematicians poaching in biological preserves; it was easy, too, for a younger generation of geneticists to read in *Mendel's Principles of Heredity* that "Mendel's conception differs fundamentally from that involved in the Law of Ancestral Heredity," and to miss what

Pearson had emphasised again and again, in papers from which I have quoted, namely the essentially statistical and descriptive character of this law. Thus a myth regarding some essential error in the biometricians' approach has persisted to this day, so that the writer of an obituary notice of Pearson could say: "Again, the rediscovery of Mendelism was a tremendous blow to the Galtonian-Pearsonian teaching of heredity and seemed almost to discredit the method upon the results of which that teaching was based."

But Pearson saw clearly, as most of his critics did not, that no theory of inheritance could discredit certain established facts following from a statistical analysis of observational data. He believed, for example, that the study of "agencies under social control that may improve the racial qualities of future generations either physically or mentally*" could be most surely advanced by a statistical attack developed on actuarial lines; further, that though a well-tested theory of individual inheritance might assist greatly in that work, a hypothesis continually subject to modification and review, as was the case with the Mendelian theory in the years following 1906, was not a reliable weapon to use in attacking the problem of practical eugenics. It was to the many aspects of this problem that Pearson was now to give more and more of his attention.

[*To be concluded*]

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APPENDIX I

SYLLABUSES OF SEVEN LECTURES DELIVERED BY KARL PEARSON AT GRESHAM COLLEGE, BASINGHALL STREET, LONDON, IN 1891 ON "THE SCOPE AND CONCEPTS OF MODERN SCIENCE"*

[These syllabuses have been reproduced because they give a very good outline of the contents of the first edition of *The Grammar of Science*, published in 1892. This summary of the fundamental concepts of science, with its detailed classification contained in the Fourth Lecture of the first Course, must be considered in the light of the scientific knowledge of that date. In the later editions of 1900 and 1911 Pearson endeavoured to keep abreast of the times; thus for example the 1911 edition (which consisted only of Part I, Physical Sciences) ended with a chapter on Modern Physical Ideas, including the Theory of Relativity as then developed.

With the rapid advance of scientific discovery it has become, however, almost impossible for a single mind to acquire that thorough grasp of the developments in widely different fields which the writer of any fundamental survey must possess. It was because Pearson felt in later years that the task of bringing his *Grammar* up to date was beyond his powers, that he would not consent to its republication although all editions were out of print.]

* The Lectures were free to the public and commenced at 6 p.m. A note at the foot of the Syllabus for the first Course reads: "Questions concerning difficulties which may arise from the Course should be given *in writing* to the Lecturer immediately after the lecture or sent by post to him at Gresham College, Basinghall Street, and they will be answered after subsequent lectures."

FIRST COURSE, HILARY TERM, 1891

Lecture I. *Tuesday, March 3rd*

THE SCOPE AND METHODS OF SCIENCE

(*a*) Geometry in its old sense an instrument for investigating physical problems. Desirability of first grasping the nature of these problems and their relation to other branches of science. Our age with its conflicting opinions and varied social experiments offers numberless problems for the individual judgment. Tendency to throw responsibility from the individual upon the State. The latter is, however, only the collective social conscience—the tribal conscience of Clifford. Clear knowledge of facts, appreciation of their sequence and relative significance, freedom of bias needed by the ideal citizen for the formation of sound judgment. This manner of forming a judgment the method of modern science. The scientific habit of mind not peculiar to the professional scientist, but the careful study of some branch of science especially helpful for its development. Modern science as a training for the mind in an exact and impartial analysis of facts is an education specially fitted to promote good citizenship. Value of science as a training depends on its method not on its material. Test of good science that it appeals in the first place to the reason. Great classics of science often intelligible without preliminary training.

(*b*) The unity of science consists alone in its method, not in its material. The value of this method for training the mind of citizens is the chief claim of science for State support. Material of science co-extensive with the whole universe, its history past and present. Great advance of science in modern times, no Humboldt could now survey its whole domain. Extension of science into regions where our great-grandfathers could see nothing or would have denied possibility of human knowledge. No *real* limit to our perception of the universe. To exclude science from any field of thought, e.g. metaphysics, is to assert that there are no facts in that field or that the laws of logical thought do not apply to them. Scientists may differ in judgment upon some groups of physical or biological facts, but they practically agree as to fundamental principles and sequences; similar agreement rapidly growing up in mental and moral sciences, though the classification of facts has not been so complete or unprejudiced. Wide divergences among writers on metaphysics and kindred subjects due to want of scientific method. That method the sole gateway to absolute knowledge or truth. Metaphysician ought to be recognised as a poet. He cannot now-a-days retard scientific research, but he too often discredits scientific method. His school of thought endeavours to wall off those fields where science is *at present* ignorant, and treats them as ground where science has no business to trespass, applying the imagination in lieu of the scientific method. Such a process does not lead to unanimity of judgment, hence it follows that science rather than philosophy is the better training for citizenship.

The cry of science: "We are ignorant" should not be interpreted as it is by some English and German scientists to mean: "We shall *always* remain ignorant." To say that the scientific method is the sole source of knowledge but that it is inapplicable in certain fields shows rash impatience or modesty akin to despair. There are two kinds of problems science confesses itself unable to solve: those in which the facts presented are (i) unreal, (ii) insufficiently classified. Examples of Astrology, Witchcraft, Alchemy of the Middle Ages, and of Spiritualism and Hypnotism to-day. Science owns its ignorance

greater than its knowledge, but refuses to have arbitrary barriers raised beyond which it shall not tread. It asserts its right to apply its method to all problems of life and mind.

(*c*) Besides providing an unsurpassed training, modern science brings clear knowledge of facts, bearing closely on social welfare, e.g. facts resumed in theories of heredity like Weismann's. The laboratory experiments of a biologist may be more important than all the philosophic theories of the State from Plato to Hegel. First demand of State upon individual is not for self-sacrifice, but for self-development. A moral judgment, i.e. one that will tend to social welfare, does not depend solely on readiness for self-sacrifice, but, in the first place, upon knowledge and method.

(*d*). Imaginative or aesthetic side of science. Great scientist in a certain sense a creative artist. Without the use of the imagination no scientific discovery. Facts being observed and classified, the creative imagination of a scientist resumes them in a single statement or law. The product of the imagination has then to be submitted to the severest criticism to test its accordance with the whole group of facts. "The first thing is to create a system, the second to be disgusted with it"—Turgot. Example in Darwin's description of his discovery of natural selection.

Resemblance between artistic and scientific creation. Our aesthetic judgment satisfied by works of art in so far as they faithfully concentrate into a single formula or a few symbols, a wide range of human feelings, facts which in a long course of experience the mind has been consciously or unconsciously classifying. Scientific interpretation of phenomena is the only one which can permanently satisfy the aesthetic judgment, which demands truth, harmony between representation and represented. Folly of saying that science destroys beauty of life. Basis of our aesthetic judgment is merely changing. Gratification of our desire for a consistent explanation of the universe is the third great claim of science on our support. This explanation really lies in the *complete* knowledge of even its most insignificant member, e.g. Tennyson's "Flower in the crannied wall."

Lecture II. *Wednesday, March 4th*

THE SCIENTIFIC LAW AND THE SCIENTIFIC FACT

(*a*) Why is scientific law true for all minds, and how does the use of the word *law* in science differ from other uses of it? Law stands to most for the rules of conduct proclaimed by the State. Austin's definition of law; it involves a command and a duty. Such a definition leaves no place for law in the scientific sense. Use of same word for two different conceptions has led to much confusion. Civil and moral law not absolute, but continually changing with social conditions. On the other hand, natural or scientific law once established, invariable and valid for all reasoning minds. Natural law preceded civil law, for the latter originated in unwritten custom, and this in instinctive habit, which is akin to natural law. This recognised by the Roman jurists in their *Lex Naturae*, and more clearly by Stoics. Stoics found the universe guided by reason; reason they termed law because it enjoins and forbids. Reason cannot be twofold, therefore reason and law in man are one with the reason and law of nature. Stoics proved too much. Human law is a natural product, but as product is continually changing and being remodelled, while the natural law by which it changes is invariable. Term law so long applied to both natural and civil law, that no change is now possible; we

can only be careful to distinguish between the two ideas. Hooker's definition of law. Positive definition of natural law. A brief statement or formula which resumes the oft-repeated sequences of a certain group of phenomena.

(*b*) Stoics and modern materialists find the source of natural law in the universe itself; those who use "the argument from design" place the source of law in a lawgiver outside the universe. A third possibility is that the reason we find in the law of nature may be put there by the only reason of which we have cognizance, the human reason. Natural phenomena as presented by our senses, may be the mirror which reflects back to us our mind. Example of the dog, who might find instinct in the universe. Newton's rule that we are not to seek for new causes to explain any group of phenomena, till all known causes have been shown to be insufficient. Human perceptive faculty considered as a great sifting machine, first unconsciously working, and then explaining its own unrecognised work. If all rational minds are instruments of the same type the sequences observed by one will hold for all. Examination of the facts and laws of science from this standpoint.

(*c*) Scientific Fact or Phenomenon. Take the blackboard; it appears a complex group of sensations. Its individuality consists in the grouping. Reality of a thing lies for us in the sameness for a long time of the majority of its properties, i.e. of our sensations of it. Senses as the sole gateways by which we obtain knowledge of things, which are synonymous for us with groups of possible sensations. Remove the latter and things cease to be. Example of needle, sight, touch, and pain. Distinction between ourselves and outside world, convenient but arbitrary; it is only a distinction between one group of sensations and another. The *facts* with which science has to deal are groups of sensations. Sensations may be analysed and reduced into simple elements which are not themselves direct subjects of sensation: e.g. molecule, atom, aether are such. The progress of science is bound up with the reduction of our sensations to more and more simple elements and the discovery of more and more comprehensive relations between their sequences. Example: the development of planetary theories, epicycles, Copernicus, Kepler, Newton.

(*d*) Law of gravitation as a good example of scientific law. It is the description of *how* every particle in the universe is changing the motion of every other particle, but it is not the *why* of this change. The universality of such a scientific law depends (i) on the world of phenomena being the same for all human minds; (ii) the reasoning powers by which the law is deduced from phenomena being valid for all minds. Normal perceptive and reasoning faculties may be machines of like type, but why should latter be able to find any law at all in sensations of former? Connection between two faculties obscure, yet in evolution of man his power of perception has apparently developed step by step with his reasoning faculty. A man with sensations having no order or law would be like an idiot without power to sift them and reason upon them; he could not survive in the struggle for existence. Madness may be a kind of atavism from a time when perceptive and reasoning faculties were not co-ordinated. The parallel evolution of the two gives a clue to the possibility we find of deciphering phenomena by rational processes. A further clue lies perhaps in the real simplicity of the channels through which the most complex sensations must reach us, i.e. the senses. Comparison of the scientific and philosophical standpoints with regard to the "reason" supposed to exist in natural law.

Lecture III. *Thursday, March 5th*

THE CONCEPTS OF SCIENCE

(*a*) Science is, as we have seen, a description of the "how" not an explanation of the "why." This view lost sight of by those, who, like Mr Gladstone, contrast the "mechanical" explanation with an "intellectual" one. Confusion arises in word *force* defined as *cause* of motion. Kirchhoff's definition of *Mechanics* (as the science of motion, its object being the complete *description* in the simplest possible manner of the motions which occur in Nature). Two ideas, resulting in confusion, attached to the word *cause*. Certain kinds of motion arising *apparently* from will of a living agent, we are apt to associate our idea of cause (final cause) with arbitrary will. This tendency of human mind exemplified by savages, by Schopenhauer, by Aristotle and mediaeval Schoolmen. Scientist when he speaks of force as a cause really describes without explaining, a change of motion, e.g. force of gravitation. A *cause* in science is any stage which invariably repeats itself in the same position in a sequence of phenomena. All stages of sequence successive causes. Thus no idea of *enforcement* or compulsion in scientific concept of cause. Two ideas of cause, an arbitrary result of will and an invariable stage in a sequence, contradict one another. Introduction of will as cause at any point in a sequence is merely a confession that at that point our knowledge ceases. Branches of science are devoted to describing the sequence of causes which lead to the determination of will. Will as cause of motion ceases to have meaning, as our ignorance with regard to motion disappears. Cause and effect like scientific law are the description of an invariable sequence and not a radical explanation of phenomena.

(*b*) Invariable order in universe means invariable order in our sensations. Without this, knowledge impossible. Man has won his dictatorship in the struggle for existence by his power of foreseeing the results of sequences, but this fact shows that his perceptive faculty must receive its sensations in invariable order. Theosophists and spiritualists who believe invariable order of phenomena can be interrupted, weaken the foundations of knowledge and so of man's supremacy.

(*c*) Invariable sequences are sequences of motion, and have therefore relation to *space* and *time*. Example of blackboard again; the "thing-in-itself" of the metaphysician; no need to discuss nor power to reach anything beyond our sensations. In order to distinguish between two groups of sensations or phenomena, we must perceive them *apart*, this "perceiving apart" is what we term co-existence in space. Space is thus a mode of perception, and is just as large as suffices to separate all phenomena which co-exist for us. Space has no size independent of the groups of sensations, we separate by means of it. Mystery of space lies in the perceptive faculty of man.

Another mode of distinguishing phenomena from each other is by their sequence, this mode of "perceiving apart" we term succession in time. Without time the only sciences possible are those dealing with position, measurement, quantity, or Algebra and Geometry, etc., where no idea of cause and effect of, or sequence occurs. Appreciation of sequence identical with consciousness or life. Time, as a mode of perception, is just so long as is needful to distinguish all the stages of the sequences, all the cause and effect known to us. Where man can no longer carry back the sequence of phenomena, there time ceases. Space and time to be looked upon as modes of perception, whose mysteries lie in the human perceptive faculty, not in an external universe. The *Exact Sciences* are those which deal with these modes of perception.

(*d*) Further analysis of concept *space*. Surfaces and lines as modes of separating groups of sensations. Fundamental concepts of Geometry found on examination to be

only ideal mental symbols of separation, not real sensations; they help us, however, to classify the latter in groups. Example—Surface and edge of Table. Discovery that not only geometrical surfaces, but all surfaces are ideal. Bodies not really *continuous*. Elasticity, etc., leads us to suppose a starlike structure. Sir William Thomson's representation of the granular constitution of a drop of water. *Atoms*, *Molecules*. Atomic Theory ending in a new continuity, that of the *Aether*. Continuous surface, atom, aether, ideas which apparently contradict each other. Really ideal concepts which enable us to distinguish and classify different phases of sensation. Progress of scientific knowledge consists in forming new or wider ideal concepts which embrace wider ranges of observed facts.

(*e*) Further analysis of concept *time*. Measurement of states of consciousness or sequence of sensations by a recurring sensation. In our case ultimately the sun on the meridian at Greenwich. But this not *absolute* time, for sun may be bad time keeper. Absolute interval of time an ideal concept of science. Newton's definition. Time as a mode of perception really peculiar to the individual. Abnormal sequences, miraculous events in time and space, connote abnormal perceptive faculty and not cataclysms in external world of phenomena. Spiritual in Natural Law.

Lecture IV. *Friday, March 6th*

THE CLASSIFICATION OF THE SCIENCES

(*a*) Any really complete classification of the sciences could only be made by a group of scientists having wide knowledge of different branches; impossible for a single man, who cannot grasp the exact bearing of each and all. Failure of single man to achieve a satisfactory classification evidenced by Mr Herbert Spencer. Criticism of his scheme; unintelligible character from the physicist's standpoint of his ideas on the "redistribution of force." Further, his divisions rather logical than practical divisions of science. Present classification pretends to no logical exactness; merely a suggestion based on views expressed in the earlier lectures, which may be summarised again thus: The subject matter of science is the complex of sensations we term phenomena. The aim of science is the expression in the simplest, and at the same time most comprehensive, formulae possible of the sequences and relations observed among phenomena.

(*b*) Starting with sensations, we may deal with (i) the forms in which sensations are presented by the perceptive faculty; (ii) the contents of those sensations. The first division is termed *Abstract Science*, it treats of the forms and ideal concepts by aid of which we observe or distinguish phenomena. Mr Herbert Spencer's remarks on the *Abstract Sciences*. Practical agreement of his view with that which considers space and time as forms of thought. The abstract sciences are sometimes spoken of as the *Exact Sciences*. Since they deal with abstract or ideal concepts, their reasoning can be perfectly valid for all that falls under these clearly and narrowly defined concepts. The second division is termed *Concrete Science*. Much of Concrete Science has not or cannot be reduced to ideal concepts admitting of exact reasoning. The distinction between abstract and concrete science perfectly clear and definite, but the subdivisions of these branches admit of no absolutely rigid boundaries, for their topics cross and recross each other, so that many sciences belong to three or more divisions at once. Hence the rough and empirical character of the following Tables. Concrete Science may for practical purposes be divided into those which deal with (i) Inorganic Phenomena, or the Physical Sciences; (ii) Organic Phenomena, or the Biological Sciences. Difficulty of defining "organic" and "inorganic." Remarks on Life and Consciousness.

A.—*ABSTRACT SCIENCE. Modes of Discrimination**.

General Relations of Discrimination			Relations Peculiar to Space and Time				
Qualitative	Quantitative		Space, Discrimination by Localisation		Time, Discrimination by Sequence		
	Discrete Quantity	Change in Quantity	Qualitative (Position)	Quantitative (Size)	Qualitative	Quantitative	
Logic. Orthology, Methodology	*Arithmetic, Algebra, Theories of Measurement, Errors, Probability, Statistics*, etc.	*Theory of Functions, Calculus of Rates or Fluxions, Calculus of Sums*, etc.	*Descriptive Geometry*	*Metrical Geometry, Trigonometry, Mensuration*, etc.	*Theories of Observation and Description* (allied to *Logic*)	*Theory of Strains* (Change in Size and Shape)	*Kinematics* (Change in Position)

* This table and the two which follow have been taken from *The Grammar of Science* (1892), pp. 458, 464 and 468, as the arrangement is somewhat more compact than that of the *Syllabus*.

B.—*CONCRETE SCIENCE. Inorganic Phenomena.*

Reduced to Ideal Motions					Not yet reduced to Ideal Motions
Precise Physical Sciences					*Synoptic Physical Sciences*
Molar Physics	Molecular Physics	Atomic Physics	Physics of the Aether		
			Apart from Molecule	In association with Molecule	
Mechanics, Planetary Theory, Lunar Theory, etc.	*Elasticity, Plasticity, Cohesion, Sound, Crystallography, Figure of the Earth, Hydromechanics, Aeromechanics, Theory of the Tides, Kinetic Theory of Gases*, etc.	*Theoretical Chemistry, Spectrum Analysis, Solar and Sidereal Physics*, etc.	*Theory of Radiation* (Light, Heat, and Electro-magnetic Waves)	*Light, Heat, Electricity, Magnetism* (in relation to Molecular Structure), *e.g.*, *Theories of Dispersion, Absorption, Transmission, Conduction*, etc.	*Nebular Theories, Evolution of Planetary System, Inorganic Evolution of the Earth. Geology, Geography* (sometimes termed *Physical Geography*), *Meteorology, Mineralogy, Chemistry*, etc.

C.—*CONCRETE SCIENCE. Organic Phenomena.*

SPACE (Localisation)	TIME (Growth or Change)												
	Non-recurring Phases *History*				Recurring Phases *Biology* { Of Plants, *Botany* / Of Animals, *Zoology*								
	General *Evolution of Species*	Special *Evolution of Man*			Form and Structure	Growth and Reproduction	Functions and Actions						
							Physical	Mental					
							Physiology	*Psychology*					
											Special in Man		General
										Group		Individual	
									Sociology				
Geographical Distribution of Living Forms (Chorology), Habits in relation to Situation and Climate (Ecology), Natural History (in old sense)	*Genesis of Life (Phylogeny, Palaeontology, etc.), Origin of Species, Theories of Natural and Sexual Selection,* etc.	Physique	Mental Faculties	Social Institutions									
	Craniology, Anthropology, etc.	*History of Language, Philology, Histories of Philosophy, Science, Literature, Art,* etc.	*Archaeology, Folklore, Histories of Customs, Marriage and Ownership of Religions, of States, of Laws,* etc.		*Morphology, Histology, Anatomy,* etc.	*Embryology, Theory of Sex, Theory of Heredity,* etc.			*Morals, Politics, Political Economy, Jurisprudence,* etc.		*Psychics, Physiology of Thought*		*Theory of Instinct, Genesis of Consciousness,* etc.

SECOND COURSE, EASTER TERM, 1891

Lecture I. *Tuesday, April 14th*

THE GEOMETRY OF MOTION

(*a*) *Résumé* of results reached in the previous course. Phenomena as groups of sensations or things capable of being felt. Scientific law as the briefest possible statement of the widest possible range of sequences among phenomena. Science as an "economy of thought." Ultimate elements of sensations we represent as motions in space and time. In same manner as Geometry dealing with position and size in space is, owing to want of continuity in bodies, an ideal mode of distinguishing sensations, so our theories of motion dealing with the change of position and size also treat of ideal concepts. The types of motion with which we deal theoretically do not occur among phenomena, but represent them so closely that they enable us to distinguish and classify a great many physical changes. As motion appears to be the fundamental factor in phenomena we have to enquire how it may be scientifically described and measured.

(*b*) The discussion of the modes in which we represent and measure motion, or change of position, forms the Geometry of Motion. Technically termed *Kinematics* (from Greek κίνημα = a movement). Motion of bodies as a whole and motion of their relative parts. Theory of strains. If we pass from the general discussion of motion to the special types which we assume owing to experiment to take place in nature, we are led to the so-called *Laws of Motion* and to the concepts *Mass and Force.*

In order to understand motion we take the simplest ideal motion we can—the motion of a point. Continuity of motion. The path, change of direction of path, slope and steepness. Shift as measuring position. Bearing, sense and size—limitations of the arithmetician and freedom of the geometrician. Steps. Parallelogram of Steps. Change of position. Galilei's curve. Idea of velocity as slope. Speed and bearing. "Hodograph." Change in velocity, spurt and shunt, technically termed Acceleration. The three factors of motion: shift, velocity and acceleration, all relative and how measured.

Definition of a "rigid" body. Its motion, shift and spin; how represented. The next stage is to consider how these concepts of motion apply to the elementary motions to which we reduce physical phenomena.

Lecture II. *Wednesday, April 15th*

MATTER AND FORCE

(*a*) Heraclitus' "All things flow" contrasted with the modern view—"All things are in motion." The three questions: What is it that moves?—Why does it move?—How does it move?—To the last only Science can give an answer.

(*b*) What is it that moves? Matter, what we know of it. Things without apparent hardness and weight move, e.g. the aether. Matter as non-matter in motion. The vortex-ring and the wrinkle in space. Forms of motion themselves move. What it is that moves really unknown to us, our sensations of its motion are only symbols, which do not, like a picture, represent it, but like words are signs we interpret by experience. Is matter a *terra incognita nunc et in aeternum*? How we base our sensations upon ideal motions, and how light may some day come through these. Atoms, how they differ from extended

bodies. Boscovich's sizeless atom. Ultimate element of matter, whether atom or not, we have no power to sensate. It must enable us to construct the mechanical universe, but cannot be itself mechanical. Matter as the point of contact between known and unknown, the doorway into a world beyond the mechanical.

(*c*) Why does matter move? We are ignorant. The physical universe as an atomic dance; the motion of each atom related to its position relative to other atoms, but why this related motion we cannot explain. Can we throw it back on the aether? Aether as a medium exerting pressure leads us again to action at a distance between ultimate parts of aether, which is as mysterious and inexplicable as the dance of atoms. Action at a distance appears a fundamental concept without any explanation being at present possible.

The *how* of motion the great field of scientific research. Its results summed up in the laws of motion. "All laws must finally be merged in laws of motion." (Helmholtz.)

Lecture III. *Thursday, April 16th*

MATTER AND FORCE (*continued*)

(*a*) The *how* of the atomic dance. The group of atoms or molecule; the group of molecules, the particle or corpuscle. Atom, molecule, and particle cannot be captured individually. Why we can reason about these ideal elements into which we resolve sensible bodies. Construction of the motion of sensible bodies from the assumed ideal motions of these insensible elements. Agreement of the motion thus constructed with experience.

(*b*) Imaginary separation of a pair of corpuscles from the rest of the universe. How they dance round and about each other, or in the usual language "influence each other's motion." Relative position determines acceleration and not velocity, the latter depends on the whole past history of any corpuscle. Influence of distance, aspect, velocities and physical conditions on the mutual accelerations. Ratio of mutual accelerations independent of the "field." Definition of *Mass*. Mass as a relative quantity or ratio of spurts. Definition of Force a convenient name for the product of mass and acceleration. Force in no sense the cause of motion, or its *why*. Muscular sense of force, really a sensation of mutual acceleration. Obscurity of the materialistic view which would "explain" the universe by matter and force. Popular notion of force as *cause* of motion a human weakness, an endeavour on our part to hide our ignorance of why bodies move.

Influence of the presence of a third corpuscle on the dance of two others. Hypothesis of modified action. Newton's Second Law. Parallelogram of Force. Bodies of uniform physical nature The relation between geometry and kinetics—mass proportional to volume. Density. Study of physical universe reduced to consideration of densities and mutual accelerations. How motion of finite bodies is constructed from assumed motion of elementary insensible corpuscles. Verification of the laws of motion. The laws of motion sufficient for the mechanical description of the universe.

(*c*) Can we never hope to learn what moves and why it moves? The most elementary type of motion and moving thing hardly yet reached in atom or aether. Possibly when the simplest mechanism of the universe is reached, this ideal concept will help us to knowledge. Possibly we must always be ignorant. Can the extensions of geometry by Riemann and Clifford help us? Flatland—possibly, out and through, in the direction of matter.

APPENDIX II

University College,
London
26th November 1904

DRAPERS' COMPANY GRANT

Dear Dr Foster,

You asked me the other day for some account of the great value that the Drapers' Grant has been to my Department and, I think, perhaps I shall best give it by writing a more or less informal letter in which it is easier to accompany the bare facts with a running commentary.

I would divide what I have to say under four headings: (i) *Personnel*, (ii) *Instruments*, (iii) *Assistance to Research* and (iv) *Publications*.

(i) *Personnel*. All the work of computing undertaken in my Department before the Drapers' Grant was made was entirely done by volunteer workers. For this reason I was never able to depend upon permanent aid, and many investigations were extended over a number of years owing to my workers being called off to other employments. This has been altered by the appointment for two years of Dr Lee as a computator, and by the occasional payment of additional calculators, e.g. Miss F. Cave, of Girton College. A sum of about £200 has or will be spent for this purpose. The result so far has been that the investigations on the inheritance of the moral and mental characters in man, which had been dragging on for several years were promptly completed, and were published as the Huxley Lecture of last year. The further investigation of the relationship between the mental and physical characters in man has made considerable progress, but as it involves the tabulation of nearly 6000 children for about 20 characters, it will be much advanced but hardly completed this year. The work really requires the undivided attention of a good computator like Dr Lee and two or three sorters for twelve months. This we have not been able to give. Our work has been interrupted over and over again by requests for statistical aid from various external bodies and individual investigators. Aid has been freely given in such cases, although that aid has often cost my staff days and weeks of work, quite unrealised by those who applied for it. A number of reports have been drawn up, which we should have had to decline doing had we not conceived that the Drapers' Grant was directly intended to assist, not only our own personal work, but scientific work in general. Among reports that have been prepared in this way I may mention:

(*a*) A report on Scottish Pauper Lunatic Rates, for one of the Scottish Lunacy Commissioners.

(*b*) A report on Time Fuses for the Royal Artillery, for one of the Woolwich officials.

(*c*) A report on the Cancer Statistics of the Cancer Research Laboratory with special reference to the Inheritance of Cancer and the Association of Tubercular with Cancerous Tendencies (published in the *Archives of the Middlesex Hospital*).

(*d*) A report on the Antityphoid Inoculation Statistics for Lieutenant-Colonel Simpson of the War Office (recently published in the *British Medical Journal*).

Further, a laborious verification of the work of Dr Greenwood of the London Hospital on the weights of the heart and other organs of the human body in health and

disease was carried out by Dr Lee; and Mr Blakeman, another assistant in the Department, has been at work for some time on the reduction of Dr Gladstone's measurements at the Middlesex Hospital on brain weight in its relation to general physique. It will be seen therefore that while our computing capacity has been called into service in a variety of ways, we have had largely in the past year to assist medical inquiries. Among those who are using or have used the Laboratory for statistical training, I may mention Major Ernest Roberts, of the Indian Medical Service (Punjaub Registrar), Dr F. Turner, of the Metropolitan Asylums Board Service, who is now at work on smallpox statistics, and Dr Greenwood, who is investigating the relationship of intelligence to size of head. I think therefore that the money spent on the computating side has been productive of very good and useful work, and that the Grant has enabled us to give to public bodies and private workers advice and assistance which they could not have procured elsewhere.

Besides those referred to above, I might add aid to Dr Macpherson of Edinburgh as to various statistical inquiries, to Mr Tocher of Aberdeen in regard to the Scottish pigmentation survey and to Mr Punnett of the Cambridge Zoological Laboratory in regard to inheritance in a species of shark.

Besides Dr Lee as computator, I have been able to procure the services of two distinguished Cambridge wranglers, first, Mr Clerk Maxwell Garnett of Trinity, and later, Mr J. Blakeman of the same College and a Master of Science of the Victoria University. This has been a most marked gain to the Department, for we have never so far been able to command the whole services of any first class man as assistant in the Department. These men have not only been able to give assistance to those working in the Department, to revise for press mathematical and other memoirs, but to solve mathematical problems arising in the course of our statistical research. The great difficulty has been that the circumstances of the Drapers' Grant does not permit of our offering hope of a permanent post to men of this type. The training they require for our statistical work is of a very special kind, which they do not come from Cambridge provided with, and when trained they are liable to be carried off to better permanent positions. One of the chief needs of my Department at University College throughout my twenty years' service has been a highly trained permanent assistant sufficiently well paid to devote the whole of his energies to the work. The Drapers' Grant has bettered this state of affairs, but only in a temporary manner, and not quite as effectively as I could have wished, as a good assistant must look to some permanency of tenure, which I cannot offer.

Lastly, I must mention Miss M. Radford, whom I was able last year to procure as a paid microscopist. She was chiefly engaged on measurements on snails, for a memoir on inheritance in snails, the data for which have been slowly accumulating for some years, the breeding experiments being at first undertaken by Dr Warren, and afterwards, when he left for South Africa, by Mr Palin Elderton. Miss Radford has also made a good many other measurements, notably a series on human teeth to compare their present character with those of prehistoric Egyptians of 8000 B.C.

On personnel about £500 of the Grant will have been expended in all.

(ii) *Instruments.* More than £110 have been spent on new apparatus, which will permanently increase the efficiency of the Department. Under this head I ought to note a Tate's Calculating Machine, the standard English Machine (£40), a very good reading Microscope (£26) for making fine measurements, a Leitz Microscope (£29), and a Coradi Parabolograph of the type designed by Mr Payne, formerly my assistant (£14). Besides the above, there have been a number of smaller pieces of apparatus, and photographic

and microscopic material. The microscopes have been of peculiar value to the Department, for we have had in the past either to borrow a microscope from another Department or apply to the Royal Society for the loan of one for each special research.

(iii) *Assistance to Research.* With regard to special material for individual researches I would note that cranial material and apparatus have been procured for Dr Macdonell's work on the English skull, an osteometer for the work on the long bones in man, studbooks have been purchased for Miss Barrington's work on inheritance in the greyhound, and the like books for Mr Blanchard's investigations on inheritance in Guernsey cattle. The memoir on greyhounds has been published, but that on the cattle involves still a year or more's work. A similar investigation is at present being made on inheritance in shorthorn cattle. Further, model dams were made for Mr Atcherley's researches into the stability of dams for water reservoirs and assistance given to Mr Andrews in his experiments on crane and coupling hooks. Fertilisation bags and other material have been provided for the different workers who have been investigating inheritance in the Shirley Poppy, a second memoir on which will shortly appear. The total expenditure under this head has been perhaps £40.

(iv) *Publications.* The amount of the Grant spent on assisting publication has fallen under two heads: (*a*) aid towards a better publication of work produced by researchers in the Department and (*b*) the special series of publications entitled "The Drapers' Research Memoirs." Under the first heading I have endeavoured to get things well printed which could not otherwise have been published effectively. Thus in the Memoir by Dr Lee and myself on the Inheritance of the Physical Characters in Man, we were able to publish the actual tables of measurement of about 1100 families, which would have had to be immensely condensed without this aid*. As it is, a record of permanent value has been issued, which, however desirable, no ordinary Journal or Editor could afford without special assistance to publish. The same remarks apply to a second piece of work—Dr Macdonell's great memoir on the English Skull. Splendidly illustrated memoirs on the skull have recently appeared in both Sweden and Germany. By aid of £35 from the Drapers' Grant and a like amount specially voted by the Council of the Royal Society, we were able to issue Dr Macdonell's memoir in a form which need fear no comparison with continental work. It stands *princeps facile* among English craniological memoirs, and will probably remain for many years the standard treatise on the English skull.

In the Drapers' Research Memoirs we have endeavoured to give expression not only to our sense of what the Drapers' Co. has done for our own Department and College, but also to the need for each technical school of academic rank in the country issuing its own series of original researches. The *Mittheilungen* of the Charlottenburg Polytechnic and of the Munich Technical School are known all the world over; the *Journal de l'École Polytechnique* is a famous French mathematico-technical Journal, but no English Technical School publishes any series of its own research memoirs. We have endeavoured to indicate the desirability of this in the Drapers' Research Memoirs *Technical Series*. Here again we have been enabled by the Grant to illustrate them in a manner which we should hardly have achieved in any other method of publication. In the third memoir of this series, for example, now at Press, there are plates of the stresses and deflections in three types of bridges reduced from the working drawings and lithographed, which we should hardly have persuaded any average Society or Journal to undertake. The same

* [The original measurements were not published, but 78 correlation tables were included in an Appendix. Ed.]

remark applies to the illustrations to the memoir on *Torsion* in axles and shafting, which we hope to publish as the fourth memoir if our funds suffice; this memoir has been waiting several years for publication, owing to there being no Journal or Society in England which will issue mathematical memoirs dealing with technical subjects, and requiring expensive illustration. Thus the Grant on the technical side has enabled us to break largely new ground, and we hope it may lead to a permanent custom on the part of the larger technical schools of publishing as in Germany and in France their own researches.

I think you will see from the above account that the Drapers' Grant has been of very great value to the workers in my Department. It has given an increased vigour and activity to every part of our work, and I can only wish that there were some means of rendering at least the more important sides of it permanent. I feel convinced that a statistical laboratory with a modest staff of computators is almost a national need at the present time. This is recognised in America, and I have recently drawn up by request of the Trustees a report for the Carnegie Institute on the possibility of establishing one in America. The need also for technical research starting from a higher scientific basis, mathematical and physical, is equally urgent. We can only endeavour to supply these needs as far as fortune puts the means in our way. The Drapers' Grant came at the right moment, when I had the right people about to give it effect. It seems a pity to bring all the agencies which that Grant has set working to an end, but as you know, I did not seek the Grant, it came as a complete surprise, and I have no power now to procure another Grant, beyond asking possible helpers to examine whether the work achieved is good or bad.

Believe me, Yours very truly,

KARL PEARSON

A STUDY OF THE LENGTHS OF THE LONG BONES OF THE ARMS AND LEGS IN MAN, WITH SPECIAL REFERENCE TO ANGLO-SAXON SKELETONS.

By Dr phil. A. HEINRICH MÜNTER.

1. *Introduction.* The following investigation deals primarily with all the Anglo-Saxon skeletons preserved in the Museum of the Royal College of Surgeons, London, in the Department of Human Anatomy of the University Museum, Oxford, in the University Museum of Anatomy, Cambridge, and in the Pitt-Rivers Museum at Farnham*. The writer acknowledges his indebtedness to the authorities of these museums for having granted him access to the material and for having aided his work in other ways†. The Anglo-Saxon collections in their charge, together with a few in other museums for which descriptions had previously been published, comprise the vast majority of those available for study anywhere, and they form a series which is long enough to define accurately the racial characters of the population represented. Detailed records relating to the skulls will be presented in a later paper. Owing to limitations of time, it was only possible to take lengths of the long bones of the arms and legs, and these are dealt with here. They relate to 233 male and 93 female adult skeletons, though few of these have all the bones considered complete enough to give the maximum lengths. The series is one of the longest of its kind for which the measurements are available, and it permits a re-examination, from a statistical point of view, of such topics as asymmetry, the variabilities, correlations and relative proportions of the limb bones, and sexual and racial differences. These topics have been discussed repeatedly by anatomists and anthropologists, but the data required in examining them are still by no means extensive. This is due partly to the difficulty of obtaining sufficiently large numbers of skeletons representing any race, but largely, also, to the fact that those who have had suitable material have not presented their measurements in forms which make a full treatment possible: individual measurements have only been given in a few cases. Most of the morphometric problems considered require, ideally, measurements of all the bones associated together in skeletons, or, failing this ideal, of paired bones for different skeletons. But series of complete skeletons are only found in dissecting rooms, and there are only two short ones from this source for which adequate measurements

* The previously unpublished measurements of the long bones in the Pitt-Rivers Museum used in this paper were kindly provided by Dr Dudley Buxton. All others in the appended tables of individual measurements were taken by the writer.

† He was able to undertake this lengthy task while receiving grants from the Academic Assistance Council and the Royal College of Surgeons, and without this aid, which is gratefully acknowledged, it could not have been carried out. The paper was written in the Galton Laboratory, University College, London. The writer feels deeply indebted to Dr G. M. Morant for all the trouble he has taken in helping him in various ways.

have been published. The records for excavated material are generally very incomplete. We have a total of 233 male and 93 female Anglo-Saxon skeletons for which all possible lengths of the humerus, radius, ulna, femur, tibia and fibula are given: there are only 4 male and 3 female skeletons having all these bones on both sides sufficiently complete to give the lengths. If we omit the ulnae and fibulae (which are most frequently broken), so that each segment of the two limbs is represented by one bone, there are 20 male and 9 female having all the remaining four bones on both sides sufficiently complete to give the lengths. To make the best use of such material, it is hence necessary to obtain different statistical constants from different sets of skeletons, which is a serious disadvantage. Some of the published series used, as far as possible, for comparative purposes are made up by bones not associated in skeletons at all, as they were confused when excavated and cannot now be associated in that way. The value of the previous series is also lessened considerably by the fact that the same technique was not used by all the workers in measuring the lengths, and there is still an urgent need for standardisation with regard to this matter. The difficulty of sexing bones anatomically must be remembered, too. It may be suggested that it should be possible to obtain better material relating to the lengths of the limbs and their segments from living people. A few living series are used below for comparative purposes, but in examining the problems here considered they are not of much value. Only one was found giving the lengths of arms and legs on both sides, and the measurements for this are not sufficiently accurate to warrant detailed statistical treatment. A more exact technique than those at present used by anthropologists in measuring the living would be required to obtain adequate measurements which might be used to examine the problems treated in this paper. The necessity for this becomes all the more evident since a general conclusion reached here is that the differences dependent on sex, side and race are much smaller than has often been supposed.

2. *Description and Treatment of the Anglo-Saxon Material.* The Anglo-Saxon skeletons measured were excavated from a considerable number of different cemeteries (given in the appended tables of individual measurements). The majority of them are pre-Christian, but there are some of Christian date, and all regions occupied by the invaders in the earlier centuries of their occupation are represented. References to the archaeological papers describing the excavations will be given in the report on the cranial material. The possibility of there being significant differences between different regional or secular divisions of the total sample available will also be examined in the same report. The only comparisons of this kind made here are those between the maximum length of the right femur for four sub-samples representing different regions (see p. 294 below), and these seem to justify the pooling of the measurements as far as treatment of the lengths of the long bones is concerned. The sexes of the skeletons given are those provided in the museum catalogues, except in a few cases previously unsexed for which the writer obtained anatomical estimates himself by examining all the bones preserved of these skeletons

All the long bones of the arms and legs on both sides were measured. Different osteometric boards were used in different museums, but these were all similar in construction and principle. The maximum lengths need not be defined separately as they were nearly all taken in the same way by obtaining the maximum separation between the fixed and movable vertical surfaces of the board making contact with opposite extremities of the bone. In determining these the bone has to be moved up and down as well as from side to side. Most definitions of these maximum measurements do not specify the up and down movement, and it is then suspected that the writer may mean the maximum length found when the bone rests on the horizontal surface of the board. The difference between this last and the true maximum, here recorded, is very small for all bones, however. The lengths taken are the true maximum of all six bones except the femur; the maximum with the bone resting on the horizontal surface of the board for the femur, and the oblique for the femur and humerus. The last two are the maximum projective lengths with the bone resting on the horizontal surface of the board and having both condyles in contact with the fixed vertical surface. These are the measurements recorded in the tables of individual measurements. Unfortunately, the true maximum length of the tibia—from the spine to the malleolus—included there is a somewhat unusual measurement, and it was discovered after the series was completed that there is little comparative material for it. For this reason two additional lengths of the bone were found, but only for the tibiae in the Royal College of Surgeons, and the individual measurements for these are not given. The other lengths are the oblique taken by introducing the spine into the orifice of the fixed vertical surface and bringing the outer parts of the condyles into contact with the surface on either side of the orifice (the bone resting on the horizontal surface of the board), and the "maximum excluding the spine" taken by introducing the spine into the orifice and applying the most prominent point of the condyles to the fixed vertical surface, and by moving the bone slightly up and down, as well as from side to side, until the maximum length to the malleolus is reached. This length is taken to either the inner or outer condyle, according to which gives the greater projection, but usually contact is made with the outer condyle.

The following means are found for the three lengths of the same tibiae at the Royal College of Surgeons:

		Male	Female
I. Maximum including spine	*R* *L*	374·5 (33) 382·6 (26)	346·0 (14) 350·8 (13)
II. Maximum excluding spine	*R* *L*	371·1 (33) 378·7 (26)	343·3 (14) 348·3 (13)
III. Oblique	*R* *L*	366·1 (33) 374·7 (26)	338·4 (14) 343·9 (13)
I—II	*R* *L*	3·4 (33) 3·9 (26)	2·7 (14) 2·5 (13)
I—III	*R* *L*	8·4 (33) 7·9 (26)	7·6 (14) 6·9 (13)

There is a close agreement between the corresponding differences for the right and left sides, so the same allowance may be made regardless of side. Where the maximum lengths excluding the spine were needed (as indicated in the tables below) the constant amounts of 4 mm. for male and 3 mm. for female bones were deducted from the maximum lengths including the spine. Where the oblique lengths were needed the constant amounts of 8 and 7 were deducted from the maximum lengths including the spine. In these cases no adjustment was required in the case of the bones in the Royal College of Surgeons as the true values were known. The indices (Table VI) are those which have been used most commonly, but some writers have used different combinations of lengths for the purpose. The angle of the femur is "the angle which the condyles make with the horizontal plane when the bone is held vertically in the 'maximum position.'" Its secant is the maximum divided by the oblique length, as described by Warren (*loc. cit. infra*, p. 153). Means, standard deviations and coefficients of variation for the lengths of all the available Anglo-Saxon long bones belonging to 233 male and 93 female skeletons are given in Table I, except that no constants of variation are

TABLE I.

Constants for Lengths of Anglo-Saxon Long Bones.

			Means		Standard Deviations		Coefficients of Variation	
			Male	Female	Male	Female	Male	Female
Femur	Maximum	*R*	463·3 ± 1·22 (153)	426·1 ± 2·05 (56)	22·4 ± ·86	22·8 ± 1·45	4·83 ± ·19	5·35 ± ·34
		L	465·7 ± 1·39 (140)	430·9 ± 1·80 (57)	24·4 ± ·98	20·1 ± 1·27	5·24 ± ·21	4·66 ± ·30
	Oblique	*R*	459·1 ± 1·26 (154)	422·1 ± 2·03 (56)	23·1 ± ·89	22·5 ± 1·43	5·03 ± ·19	5·33 ± ·34
		L	462·7 ± 1·35 (139)	426·7 ± 1·96 (58)	23·6 ± ·95	22·1 ± 1·38	5·10 ± ·21	5·18 ± ·33
Tibia	Maximum*	*R*	378·9 ± 1·46 (103)	350·4 ± 1·99 (44)	22·0 ± 1·03	19·6 ± 1·41	5·81 ± ·27	5·59 ± ·40
		L	383·4 ± 1·54 (114)	351·9 ± 1·72 (49)	24·3 ± 1·09	17·8 ± 1·21	6·34 ± ·28	5·06 ± ·35
Fibula	Maximum	*R*	366·2 ± 2·55 (32)	332·1 (16)	21·4 ± 1·80	—	5·84 ± ·49	—
		L	362·0 ± 2·72 (32)	337·7 (9)	22·8 ± 1·92	—	6·30 ± ·53	—
Humerus	Maximum	*R*	337·1 ± 1·15 (121)	312·5 ± 1·54 (47)	18·7 ± ·81	15·7 ± 1·09	5·55 ± ·24	5·02 ± ·35
		L	333·1 ± 1·07 (105)	309·6 ± 1·35 (46)	16·2 ± ·75	13·6 ± ·96	4·86 ± ·21	4·39 ± ·31
	Oblique	*R*	330·0 ± 1·14 (115)	307·9 ± 1·44 (47)	18·2 ± ·81	14·6 ± 1·02	5·52 ± ·24	4·74 ± ·33
		L	326·2 ± 1·05 (100)	304·1 ± 1·30 (48)	15·5 ± ·74	13·4 ± ·92	4·75 ± ·23	4·41 ± ·30
Radius	Maximum	*R*	251·6 ± 1·12 (79)	227·6 ± 1·41 (34)	14·8 ± ·79	12·2 ± ·99	5·88 ± ·32	5·36 ± ·44
		L	251·9 ± 1·14 (67)	227·6 ± 1·31 (34)	13·8 ± ·80	11·3 ± ·92	5·48 ± ·29	4·96 ± ·41
Ulna	Maximum	*R*	274·2 ± 1·48 (61)	247·2 ± 1·83 (25)	17·1 ± 1·04	13·6 ± 1·30	6·24 ± ·38	5·50 ± ·53
		L	273·8 ± 1·29 (59)	248·4 ± 1·47 (24)	14·7 ± ·91	10·7 ± 1·04	5·37 ± ·33	4·31 ± ·42

* Maximum including spine.

N.B. In this and other tables the figures following the ± signs are probable errors.

given for the female fibulae as the numbers available are too small. Comparisons may be made between different groups of these constants, and in doing this we shall omit those for the fibula. Making sexual comparisons first, the sex ratios (male mean/female mean) were calculated for the 14 lengths and found to range from 1·072 to 1·109. Corresponding values for the right and left sides are extremely close, the maximum divergence between such pairs found being ·009, and the differences between the sex ratios for different bones are so small that no significance can be attached to them. The male standard deviations are greater than the female in 13 out of the 14 comparisons, and in the remaining case (Femur max. *R*) the two are almost identical. Only one of the differences exceeds 3·5 times its probable error (viz. that for Tibia max. L: $\Delta/(\text{p.e. } \Delta) = 4{\cdot}0$), but there is no doubt that more significant differences would be found for larger numbers of bones. The male coefficients of variation exceed the female in 11 cases out of the 14, but no one of the differences exceeds 3·5 times its probable error. The male bones thus appear to be both absolutely and relatively more variable than the female, but the sex difference is greater for absolute than for relative variability.

Ideal comparisons between constants for the right and left sides cannot be made for the data in the table. In doing this for a particular length for one sex it would clearly be more exact to use only those measurements which are available for both sides, and then the bilateral correlations would have to be taken into account in estimating the significance of the differences between the bilateral constants, and these correlations are known to be very high. For our material, however, this selection would reduce the numbers so much that no detailed treatment would be profitable, and accordingly somewhat inexact bilateral comparisons will be made between the data given in Table I*. Neglecting the allowance which should be made for the bilateral correlations, no one of the differences between the corresponding means for the right and left sides can be considered significant, but if the correlations could be taken into account some of them would doubtless prove to be so. For both sexes the mean lengths of the right humerus exceed those of the left, while for the femur and tibia the left bones are, on the average, the longer, all the differences in these cases being significant in all probability. For the radius and ulna, however, there is no clear suggestion that the means for one side differ from those for the other to a significant extent. The standard deviations and coefficients of variation show relations different from the means. For both males and females these constants for the right arm bones are greater than the corresponding values for the left, and most of the differences would probably be found to be significant if allowance could be made for the bilateral correlations. For the oblique length of the femur there seem to be no appreciable bilateral differences in variability, and for the maximum lengths of the femur and tibia differences are found which are probably significant, though they are of different signs for the two sexes.

* Some bilateral differences of means for the same Anglo-Saxon skeletons are considered below, pp. 273—275.

Comparisons may also be made between the constants in Table I relating to different bones, or different measurements of the same bone, a male value only being compared with another male value, or a female with a female. Here again correlations—viz. those between the lengths of different bones, or the different lengths of the same bone—should be taken into account, which would require, ideally, that only those skeletons for which all the bones are available should be used. Again we will neglect the correlations and make somewhat inexact comparisons. It will be seen that for both males and females there is a high, but not perfect, correlation between the means and standard deviations. The coefficients of variation tend to be closely similar for the different bones, the male range being from 4·75 to 6·34 and the female from 4·31 to 5·59. There is no suggestion that relative variation is significantly different for the oblique and maximum lengths of the same bone and on the same side, and no clear suggestion that any one bone is appreciably more variable than any other. The mean of the 8 male coefficients of variation relating to arm bones is 5·46 and of the 6 male values (omitting those for the fibula) relating to leg bones 5·39. The mean of the 8 female coefficients relating to arm bones is 4·84 and of the 6 female values relating to leg bones 5·20. The sexual difference in these constants has already been noted, but for the same sex there appears to be a remarkable constancy in the relative variability of different bones.

Tables II—V give correlations between lengths of the male long bones for the Anglo-Saxon series. The numbers of individuals on which they are based range from 45 to 152, and several with the ulna are omitted because they can only be found for fewer than 45 individuals. For the same reason, no correlations with the fibula and no female values were calculated except the four given in the footnotes to Table II. Considering the data in this table first, there are six correlations in it between the same measurement on the right and left sides and the extremes of their range—from ·9835 ± ·0021 to ·9592 ± ·0071—do not differ significantly. The three bilateral correlations for leg bones are actually all higher than those for arm bones, but the differences are so small that little significance can be attached to this fact. In the case of the femur the highest correlations are for the maximum and oblique lengths on the same side, the bilateral correlations follow next and the lowest are for a maximum length on one side and an oblique on the other. Most differences between pairs of the femur correlations are quite insignificant, but the greatest difference found—viz. that between ·9936 ± ·0007 for max. *L* and obl. *L* and ·9654 ± ·0045 for max. *L* and obl. *R*—is 6·2 times its probable error. Exactly the same sequence for the types of correlations is found for the humerus and the difference between the extreme values in this case is 3·7 times its probable error. At the same time, every femur correlation is greater than the corresponding humerus correlation and, taking them in the order given in the table, the first two of the differences of this kind are insignificant, the third is 4·8 times its probable error, the fourth 5·1 times, the fifth 3·8 times and the last insignificant. There is a clear tendency in the case of the correlations considered for the lengths of the leg bones to be more closely associated than the lengths of the arm bones. The few

TABLE II.

Correlations between Lengths of the same Bone (unilateral and bilateral) for Anglo-Saxon Male Skeletons.

Femur	Max. *R* and Max. *L*	·9835 ± ·0021	(106)
	Obl. *R* and Obl. *L*	·9814 ± ·0024	(106)
	Max. *R* and Obl. *R*	·9915 ± ·0009†	(152)
	Max. *L* and Obl. *L*	·9936 ± ·0007‡	(138)
	Max. *R* and Obl. *L*	·9797 ± ·0027	(104)
	Max. *L* and Obl. *R*	·9654 ± ·0045	(106)
Tibia	Max.* *R* and Max.* *L*	·9811 ± ·0030	(69)
Humerus	Max. *R* and Max. *L*	·9747 ± ·0042	(64)
	Obl. *R* and Obl. *L*	·9592 ± ·0071	(58)
	Max. *R* and Obl. *R*	·9768 ± ·0029§	(113)
	Max. *L* and Obl. *L*	·9790 ± ·0028‖	(100)
	Max. *R* and Obl. *L*	·9415 ± ·0098	(61)
	Max. *L* and Obl. *R*	·9637 ± ·0062	(60)
Radius	Max. *R* and Max. *L*	·9702 ± ·0059	(45)

* Maximum including spine.
† The female coefficient corresponding to this is ·9929 ± ·0013 for 56 bones.
‡ The female coefficient corresponding to this is ·9952 ± ·0009 for 56 bones.
§ The female coefficient corresponding to this is ·9861 ± ·0027 for 47 bones.
‖ The female coefficient corresponding to this is ·9808 ± ·0038 for 46 bones.

TABLE III.

Correlations between the Lengths of different Bones for Anglo-Saxon Male Skeletons: Leg Bone with Leg Bone and Arm Bone with Arm Bone.

	Femur Max. *R*	Femur Max. *L*	Femur Obl. *R*	Femur Obl. *L*
Tibia Max.* *R*	·8655 ± ·0177 91	·8816 ± ·0176 73	·8694 ± ·0172 92	·8631 ± ·0201 73
Tibia Max.* *L*	·8694 ± ·0176 88	·8881 ± ·0143 99	·8649 ± ·0179 90	·8746 ± ·0159 99
	Humerus Max. *R*	**Humerus Max. *L***	**Humerus Obl. *R***	**Humerus Obl. *L***
Radius Max. *R*	·8969 ± ·0162 66	·8866 ± ·0200 52	·8842 ± ·0192 59	·8718 ± ·0241 45
Radius Max. *L*	·8564 ± ·0252 51	·8269 ± ·0299 51	·7953 ± ·0365 46	·7788 ± ·0383 48

* Maximum including spine.

female correlations show differences from the corresponding male values which are quite insignificant*.

Table III gives two groups of correlations, the first being for pairs of lengths of different leg bones and the second for pairs of lengths of different arm bones. The maximum differences found within each group are quite insignificant, which is an interesting point. There is no evidence that the correlation between the femur length and tibia length on the same side is more intense than that between the femur length on one side and the tibia length on the other, and the same is observed for the humerus and radius in place of the leg bones. The mean of the 8 correlations for leg bones is ·8721 and that of the 8 for arm bones ·8496. There is far less evidence here of the tendency (suggested by the data in the previous table) for the lengths of leg bones to be more closely associated than the lengths of arm bones.

TABLE IV.

Correlations between the Lengths of different Bones for Anglo-Saxon Male Skeletons: Leg Bone with Arm Bone.

	Femur Max. *R*	Femur Max. *L*	Femur Obl. *R*	Femur Obl. *L*	Tibia Max.* *R*	Tibia Max.* *L*
Humerus Max. *R*	·8775 ± ·0156 99	·8629 ± ·0190 82	·8762 ± ·0157 100	·8665 ± ·0188 80	·8702 ± ·0193 72	·8405 ± ·0238 69
Humerus Max. *L*	·8284 ± ·0246 74	·8125 ± ·0260 78	·8432 ± ·0225 75	·8532 ± ·0209 77	·8062 ± ·0313 57	·8176 ± ·0277 65
Humerus Obl. *R*	·8454 ± ·0197 95	·8470 ± ·0216 78	·8668 ± ·0171 96	·8499 ± ·0212 78	·8340 ± ·0247 69	·8339 ± ·0255 65
Humerus Obl. *L*	·7864 ± ·0301 73	·8069 ± ·0270 76	·8171 ± ·0261 74	·8474 ± ·0220 75	·7643 ± ·0378 55	·8131 ± ·0288 63
Radius Max. *R*	·8358 ± ·0260 61	·8023 ± ·0316 58	·8590 ± ·0225 62	·7686 ± ·0366 57	·8065 ± ·0321 54	·8620 ± ·0240 52
Radius Max. *L*	·7902 ± ·0336 57	·7937 ± ·0331 57	·7812 ± ·0348 57	·7724 ± ·0360 57	·7924 ± ·0355 50	·8303 ± ·0280 56

* Maximum including spine.

A remarkable constancy is also found for the correlations in Table IV, which are between lengths of a leg bone and an arm bone. The range of correlations is from ·8775 to ·7643 and the difference between these extremes is insignificant. The following means are found for different groups of the total 36 correlations:

Total ·8267 (36)

R bone with *R* bone ·8524 (9)
L „ „ *L* „ ·8163 (9)
R „ „ *L* „ ·8191 (18)

Humerus and Femur ·8430 (16)
„ and Tibia ·8225 (8)
Radius and Femur ·8004 (8)
„ and Tibia ·8228 (4)

* See Editorial Note on p. 294 regarding the significance of differences between high correlation coefficients.

There is a clear suggestion that the right-right correlations tend to be greater than the left-left, and for corresponding pairs in Table IV they are so in 8 cases out of 9. It may be noted, too, that for the correlations in Table III the right-right correlation exceeds the corresponding left-left in 2 cases out of 4, and for the two male and two female comparisons of this kind between the constants in Table II the left-left correlations are the greater in 3 cases. All we can conclude is that there may be a tendency for correlations between the lengths of two bones of the right side to be slightly greater than the corresponding values on the left. Returning to the consideration of the data in Table IV, the mean correlations given above for different pairs of bones are so close that it would be unwise to attach any significance to the differences between them, but it may be noted that the upper arm and upper leg bones, on the one hand, and the lower arm and lower leg bones, on the other, show slightly greater means than combinations between the upper bone of one limb and the lower bone of the other.

Correlations between the lengths of the ulna and the other long bones are given separately in Table V. Some are omitted because they could only be given for samples made up by fewer than 45 individuals. All the correlations considered above are between bones of different segments of the limbs, but those for a length of the ulna with a length of the radius are for two bones in the same segment. These are the highest values in Table V and they are almost exactly

TABLE V.

Correlations between the Lengths of the Ulna and other Long Bones for Anglo-Saxon Male Skeletons.*

	Ulna Max. *R*	Ulna Max. *L*
Femur Max. *R*	·8494 ± ·0247 (58)	·8540 ± ·0248 (54)
Femur Max. *L*	·8869 ± ·0204 (50)	·8058 ± ·0316 (56)
Femur Obl. *R*	·8521 ± ·0254 (53)	·8581 ± ·0247 (52)
Femur Obl. *L*	—	·8617 ± ·0241 (52)
Tibia Max.† *R*	·8259 ± ·0306 (49)	·8690 ± ·0151 (50)
Tibia Max.† *L*	·8821 ± ·0212 (50)	·8963 ± ·0182 (53)
Humerus Max. *R*	·8267 ± ·0293 (53)	—
Humerus Obl. *R*	·8823 ± ·0213 (49)	—
Radius Max. *R*	·9766 ± ·0044 (50)	—
Radius Max. *L*	—	·9688 ± ·0060 (47)

* Only those correlations which can be based on 45 or more skeletons were calculated.

† Maximum including spine.

equal to the bilateral correlation for the radius (Table II), that for the ulna being omitted for the reason stated. The remaining correlations in the table show no significant differences from one another and, in particular, those for the ulna and humerus are not significantly greater than those between the ulna and a leg bone. The mean of the 11 values of this last kind is ·8583, which is only slightly higher than the mean for the 36 in Table IV.

The conclusions to be drawn from these data are that any differences in correlation there may be dependent on side, sex and position (lower or upper) in a member are very small, and no particular bone tends to exhibit higher correlations than any other. Bilateral correlations are highest (apart from those for two different measurements of the same bone on the same side), and of the same order as those for two bones of the same segment (judging from the scanty data for the radius and ulna); those between two bones of different segments of the same member are distinctly lower, but next in order, and last of all, but still high and very close to the correlations of the last kind, are those between the lengths of bones of different members.

TABLE VI.

Constants for Indices and an Angle derived from Lengths of the Anglo-Saxon Long Bones.

		Means		Standard Deviations	
		Male	Female	Male	Female
$100 \times \frac{\text{Radius Max.}}{\text{Humerus Max.}}$	*R*	74·6 ± ·19 (62)	73·5 ± ·22 (31)	2·18 ± ·13	1·84 ± ·16
	L	75·4 ± ·23 (48)	74·2 ± ·21 (26)	2·40 ± ·17	1·62 ± ·15
$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$	*R*	73·5 ± ·14 (100)	73·6 ± ·25 (36)	2·05 ± ·10	2·21 ± ·18
	L	72·2 ± ·15 (77)	72·7 ± ·22 (36)	1·89 ± ·10	1·94 ± ·15
$100 \times \frac{\text{Tibia Obl.*}}{\text{Femur Obl.}}$	*R*	81·1 ± ·17 (92)	80·8 ± ·20 (38)	2·43 ± ·12	1·79 ± ·14
	L	81·0 ± ·18 (99)	80·9 ± ·23 (40)	2·61 ± ·13	2·17 ± ·16
$100 \times \frac{\text{(Humerus Max.} + \text{Radius Max.)*}}{\text{(Femur Obl.} + \text{Tibia Max.)}}$	*R*	70·7 ± ·16 (41)	70·6 ± ·21 (21)	1·49 ± ·11	1·46 ± ·15
	L	69·8 ± ·16 (35)	69·8 ± ·23 (17)	1·39 ± ·11	1·39 ± ·16
Angle of Femur†	*R*	6°·9 ± ·13 (152)	7°·5 ± ·19 (56)	2°·35 ± ·09	2°·08 ± ·13
	L	6°·5 ± ·13 (138)	7°·7 ± ·21 (56)	2°·22 ± ·09	2°·30 ± ·15

* The oblique and maximum (excluding spine) lengths of the tibia used here were found directly only for the bones in the Royal College of Surgeons, fixed allowances being used to estimate them in the case of all the other tibiae: see p. 260 above.

† For definition see p. 261.

Table VI gives means and standard deviations for the four indices derived from the lengths of different long bones and for the angle of the femur. The numbers on which the female constants can be based are particularly small. There is no suggestion of a sexual differentiation of mean values except in the case of the radius-humerus index (the male value exceeding the female by an amount which is 3·8 times its probable error for the right side and 3·9 times for the left), and of the angle of the femur (the female value exceeding the male by an amount

which is 2·6 times its probable error for the right side and 4·9 times for the left). No differences between corresponding male and female standard deviations in excess of 3·5 times their probable errors are found, but for six of the ten comparisons the male constant is the greater. It is probable that some more significant sexual differences would be found if these constants could be given for larger series. It may be noted that for both males and females the means of humerus-femur and inter-membral indices are greater on the right side than on the left, the female bilateral differences for these being insignificant and the male differences being 6·3 and 4·0 times their probable errors, respectively. There is no suggestion of a significant bilateral difference in the case of the other means. The bilateral differences of standard deviations are all quite insignificant. Only one correlation between measurements of shape was found. For the male radius-humerus and tibia-femur indices on the right side the correlation for 41 skeletons is $+\cdot064 \pm \cdot105$.

The question of the estimation of the average statures for male and female Anglo-Saxons from the lengths of their long bones may now be considered. The formulae given by the late Professor Karl Pearson in 1898 were used for this purpose*. These are based on measurements of French cadavers, but no formulae which could be used more appropriately have been given since. Measurements of the lengths of the bones of the right side are used. For femur, humerus and radius these are the maximum lengths, but for the tibia the oblique length is required. This last measurement was only found for the bones at the Royal College of Surgeons, and for the others it has to be estimated by applying the corrections given on p. 261 above. Owing to this fact, the reconstruction formulae involving the tibia must be supposed less reliable than the others. The reconstructed statures obtained are given in Table VII, and in computing these the means based on the total numbers available in the case of each bone were used†. In the case of a formula involving more than one bone, the number used for the probable error was the smaller of two numbers, or the smallest of the four numbers in the case of the last formula. The reconstructed male statures are all in close agreement except the one obtained from the length of the tibia alone. The mean of all ten is only slightly less than the mean of the six which do not involve the tibia. For the females the leg bones give appreciably lower reconstructed statures than the arm bones, but it must be remembered that the numbers are all smaller. It will be safe to conclude that the best estimates of the average statures of the Anglo-Saxon population which can be given at present are close to 1683 mm. for men and 1568 mm. for women. The resulting sex ratio (1·073) is not unusual. Dr John Beddoe collected maximum lengths of Anglo-Saxon femora previously published and relating to "about 50 men and 25 women‡." His means, turned

* "Mathematical Contributions to the Theory of Evolution. V. On the Reconstruction of the Stature of Prehistoric Races," *Philosophical Transactions of the Royal Society of London*, Vol. CXCII. A (1898), pp. 169—244.

† All four lengths on the right side are only available for 46 male and 19 female skeletons.

‡ "On the Stature of the older Races of England, as estimated from the Long Bones," *The Journal of the Anthropological Institute*, Vol. XVII. (1887), pp. 202—209.

into mm., and without distinction of right and left sides, are 471·7 for males and 427·7 for females. The first of these values is surprisingly larger than the male mean in our Table VII, and several of the bones used are common to both series. Beddoe gave reconstructed statures of 1747 for men and 1602 for women. Using the same means, but his own more accurate formulae, Professor Pearson reduced

TABLE VII.

The Reconstruction of Living Stature from Lengths of different Anglo-Saxon Long Bones (Pearson Formulae).

		Male	Female
		Means*	
Femur Max. *R* (*F*)		463·5 (161)	425·7 (59)
Tibia Obl. *R* (*T*)†		370·6 (111)	343·5 (47)
Humerus Max. *R* (*H*)		337·1 (129)	312·3 (52)
Radius Max. *R* (*Ra*)		251·6 (86)	228·1 (36)
Formula	Bones	Reconstructed Stature	
a	*F*	1684·4 ± 1·7	1556·4 ± 2·8
b	*H*	1681·8 ± 1·9	1574·9 ± 3·2
c	*T*	1667·9 ± 2·2	1555·7 ± 2·5
d	*Ra*	1682·2 ± 2·9	1574·7 ± 4·5
e	*F* + *T*	1679·4 ± 1·9	1557·7 ± 2·3
f	*F*, *T*	1680·1 ± 1·9	1557·6 ± 2·3
g	*H* + *Ra*	1686·9 ± 2·4	1578·9 ± 4·0
h	*H*, *Ra*	1680·2 ± 2·4	1575·9 ± 3·9
i	*F*, *H*	1686·2 ± 1·7	1565·1 ± 2·9
k	*F*, *T*, *H*, *Ra*	1681·9 ± 2·1	1560·9 ± 3·4
Mean of all		1681·1	1565·6
Mean of *a*, *b*, *d*, *g*, *h*, *i*		1683·6	1571·0

* The Farnham bones were included in calculating the means in this table, but they were omitted in finding the means for the same lengths in Table I.

† The oblique length of the tibia was taken only for the bones at the Royal College of Surgeons, a fixed allowance being made in all other cases to determine it from the maximum length including the spine: see p. 261 above.

these estimates to 1700 and 1560, respectively: this male value is still very significantly higher than our estimate for a much longer series, but the female values are close. It may be noted that a mean male stature of 1683 is below that of the general population of Englishmen to-day, which is close to 1700, but it is well above the inter-racial mean for all modern European races.

3. *Comparative Material.* Measurements of the long bones of series of skeletons representing different races are given in the following sources (*a*)—(*k*). These are believed to include all the longer published series which are presented in such a way that the data for them are of any statistical value. Unless otherwise stated, individual measurements, standard deviations and correlations are not provided. Data relating to living series, taken from the sources (*l*)—(*o*), are used for some purposes in the following section of this paper.

(*a*) *American Indian.* Aleš Hrdlička: "The Principal Dimensions, Absolute and Relative, of the Humerus in the White Race." *American Journal of Physical Anthropology*, Vol. XVI. (1932), pp. 431—450. Mean measurements of some of the other long bones, besides the humerus, are provided in this paper for a number of male and female series. The sides are considered separately, and the data relate to paired bones. The Indian series is merely said to represent "full-blood American Indians," and the origin of the skeletons is not specified. In another paper ((*f*) below) the same writer gives a few means for a longer series of American Indian long bones, but no use has been made of these as the sides are not separated. See "Anthropometry," *Ibid.*, Vol. III. (1920), pp. 147—173, for definitions of Dr Hrdlička's long bone measurements.

(*b*) *American Indian: Pecos Pueblo.* Earnest Albert Hooton: *The Indians of Pecos Pueblo: a Study of their Skeletal Remains.* New Haven, 1930. Means, standard deviations and coefficients of variation are given for male and female series and the sides separately, but not for complete skeletons or paired bones. These are for a number of sub-samples and for the total series, the latter constants being the only ones considered here. The skeletons are of different dates, the majority being pre-Conquest, and it is improbable that any "are much more than one thousand years old." No definitions of the measurements appear to be given, but, except in the case of the tibia, there is little doubt as to the ways in which they were taken.

(*c*) *American Negro.* Aleš Hrdlička: *ioc. cit.* (1932). The skeletons represent "the general run of the American negro of the time, full-blood and mixed combined." The majority of the data for this series given below were obtained from means provided in the 1932 paper, but the indices for it (in our Table XV) are from Dr Hrdlička's 1930 paper ((*f*) below).

(*d*) *Chinese.* Paul Huston Stevenson: "On Racial Differences in Stature Long Bone Regression Formulae, with special reference to Stature Reconstruction Formulae for the Chinese." *Biometrika*, Vol. XXI. (1929), pp. 303—318. Individual measurements are given of the lengths of the four principal bones of the right side only for 48 modern male Chinese skeletons collected at Peiping. All the correlations between pairs of these lengths are provided, and they are quoted below. The measurements are those used in the Pearson formulae for reconstructing stature, viz. the maximum of the femur, humerus and radius, and the oblique of the tibia.

(*e*) *Egyptian* (*Predynastic*). Ernest Warren: "An Investigation on the Variability of the Human Skeleton: with especial reference to the Naqada Race discovered by Professor Flinders Petrie in his Explorations in Egypt." *Philosophical Transactions of the Royal Society of London*, Vol. CLXXXIX. B (1897), pp. 135—227. Individual measurements and a number of the usual statistical constants (including some correlations) are given for the lengths and other measurements of the series of late predynastic skeletons from Naqada. There are few skeletons providing lengths of all the limb bones. Some constants are quoted below from Warren's paper, but the majority for the Egyptian series given here were calculated by the present writer from the individual measurements.

(*f*) *Eskimo.* Aleš Hrdlička: "Anthropological Survey in Alaska." *Forty-Sixth Annual Report of the Bureau of American Ethnology*, 1928—1929 (1930), pp. 19—374. Mean measurements of the lengths and indices of the long bones of Western Eskimo (Alaskan) series are given. The sexes are dealt with separately, but all means are for the bones of the right and left sides taken together. The Western Eskimo indices in our Table XV are taken from the table on p. 316 of the *Report*; the "Western Eskimo" sex ratios in our Table IX are for the "Southwestern and Midwestern groups" of the table on p. 314 of the *Report*.

(*g*) *European* (various series). The means for these are given by Hrdlička in his 1932 paper, *loc. cit.* (*a*). The series—presumably all modern—are those given in our Table VIII, and it will be noted that two of the groups are miscellaneous in character. The "four European" series in our Table IX are the Irish, German, Italian and "other white."

(*h*) *French.* Etienne Rollet: *De la Mensuration des Os Longs des Membres* (*Bibliothèque d'Anthropologie Criminelle et des Sciences Pénales*), 1888. Individual measurements are given of all the long bones of 50 male and 50 female French cadavers. A few of the measurements are missing for reasons which are not stated. Statistical constants for this series were given by Miss Lee and Professor Karl Pearson in 1897 (*Proceedings of the Royal Society*, Vol. LXI. pp. 343—357) for the bones of the right side only, except that the length of the left bone was used in cases where that of the right is missing, thus giving all constants for 50 individuals. The latter author used the same practice in finding reconstruction formulae for stature from the same series (*Philosophical Transactions of the Royal Society of London*, Vol. CXCII. A (1898), pp. 169—244). For the purpose of the present paper it appeared better to keep measurements of the two sides entirely separate, and hence all the constants given below for the French series were obtained directly from Rollet's tables. It is important to note that the length of the tibia given is the oblique (see Rollet's paper, p. 10, and p. 260 above), as this is not clear from some of the papers in which Rollet's material is used. In a few cases the oblique length of the femur given is greater than the maximum, which is impossible, and for these the two measurements were interchanged.

(*i*) *Lapp.* K. E. Schreiner: "Zur Osteologie der Lappen." 1 Band. *Instituttet for Sammenlignende Kulturforskning.* Serie B: *Skrifter*, XVIII. 1 (Oslo, 1935).

Means and measures of variability are given for male and female series of skeletons (many being incomplete) from a number of Lapp cemeteries in Finland, the majority of which were used in the eighteenth century. The constants are provided for four different groups and for the total sample, only those for the last being quoted below. The majority of them relate to the right and left sides separately, but a few (including those for indices) are for the two sides together. Martin's technique of measurement is followed.

(*j*) *Norwegian.* K. Wagner: "Mittelalter-Knochen aus Oslo, eine Untersuchung 3534 langer Extremitätenknochen nebst 73 ganzer Skelette." *Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo.* I. Mat. Naturv. Klasse, 1926, No. 7 (1927). Means and measures of variability are given for male and female series of medieval bones (not associated together in skeletons) from two churches and church-yards in Oslo. Another series of bones associated together in skeletons is also described, but we have not used this on account of its small size. The bones of the right and left sides are dealt with separately, but only those for both sides together are quoted in our Table XV, as the numbers for unassociated bones would be too small. Martin's technique is followed.

(*k*) *Swiss: Alamannen.* Franz Schwerz: "Morphologische Untersuchungen der Extremitätenknochen aus dem alamannischen Gräberfelde von Augst (Kanton Aargau)," and "Untersuchungen über die Körpergrösse, die Proportionsverhältnisse und die Symmetrie der Skelette aus dem alamannischen Gräberfelde von Augst (Kanton Aargau)." *Archiv für Anthropologie.* Neue Folge, Band XV. (1917), pp. 121—141 and 196—205. Mean lengths are given for the two sides separately (without distinction of sex), and for the two sexes separately (without distinction of side): we are only able to use the latter. For the indices, in the second paper, means and distributions are given for the sexes separately without distinction of sides: standard deviations were calculated from the distributions. The measurements are insufficiently defined, but it is probable that Martin's definitions were followed.

The four following series of measurements of living people were used in order to make a few comparisons given in the text below:

(*l*) *English.* Karl Pearson: "Variation in Man and Woman." *The Chances of Death and other Studies in Evolution,* Vol. I. (1897), pp. 256—377. Means and coefficients of variation for stature, span and length of left forearm for 1000 men and 1000 women (pp. 294, 299 and 304) are used below. The statistics are for the "Family Measurement Data" collected by Professor Pearson.

(*m*) *German.* Adalbert Thiele: *Über die Längen- und Dickenunterschiede an den Extremitäten Rechter- und Linkerseits.* Inaugural-Dissertation, Würzburg (1893). Individual measurements are given for 100 youths and men of the German working classes. The paper is of interest as it is the only one we have been able to find giving individual measurements of the lengths of the arms and legs on both sides. Excluding all under 20 years of age, the distributions of the bilateral

differences for the remaining 71 men were found. Unfortunately, it was clear from these that a zero difference was recorded with far greater frequency than any which could be supposed due to chance. This defect renders the material practically worthless for statistical purposes, and the only figures for it given below are the bilateral differences of the means, while these are of uncertain value.

(*n*) *German: Fehmarn Island.* K. Saller: "Die Fehmaraner. Eine anthropologische Untersuchung aus Ostholstein." *Deutsche Rassenkunde*, Band IV, Jena, 1930. Means and constants of variability are given for different age groups of male and female inhabitants of Fehmarn Island. A few of these constants for the largest (25—60 years) group were used, in the case of stature (Saller, p. 20), length of right arm (p. 42) and span (p. 46).

(*o*) *Korean.* S. Arase, M. Kohama, G. Shima, T. Nishioka, H. Tanabe, I. Takamure and T. Kawaguchi: "Physisch-anthropologische Untersuchungen an Koreanern, I. u. II. Mitteilungen." *The Journal of the Chosen Medical Association*, Vol. XXIV. (1934), pp. 59—154. (In Japanese with short résumé in German.) The same constants are given for thirteen groups of male and female Koreans representing the inhabitants of all parts of the country. Only the six series for which both male and female means are based on 55 or more individuals have been used in the case of stature (*Report*, p. 112), span (p. 113), length of right leg (pp. 113—114) and length of right arm (p. 115).

4. *Comparisons of the Means, Constants of Variability and Correlations for Lengths of the Long Bones in various Racial Series.* The results of comparisons similar to those made for the Anglo-Saxon material alone may now be given for the comparative material listed above. The differences found between the mean lengths of the long bones for the right and left sides are given in Table VIII—the fibula and ulna being omitted as the data for them are scanty—and the numbers in brackets are the numbers of *skeletons* involved. Several of the series for which long bone measurements have been published do not appear in the table because the means for them are only given for one side, or for the two sides together, or else because the means for the right and left side are not based on precisely the same set of skeletons, while measurements for individual bones are not given. The need must be stressed of using great caution in compiling data of this kind. It might be thought, for example, that if the mean lengths of the femur, say, are given for 100 right and 100 left bones, while there were 70 skeletons having both femora and 60 others having one only, that the difference of these means would give a sufficiently close approximation to that which would be obtained from 100 complete skeletons. This is unlikely to be so, however, as can be demonstrated in the case of the Anglo-Saxon male material. Using all the bones available, the bilateral differences ($R—L$) of the mean lengths (given in Table I) are

Femur (oblique)	Tibia	Humerus	Radius
−3·6	−4·5	+4·0	−0·3

TABLE VIII.

Differences between the Mean Lengths of Long Bones of the Right and Left Sides (R—L). Paired Bones.*

Series	Author	Male				Female			
		Femur	Tibia	Humerus	Radius	Femur	Tibia	Humerus	Radius
Anglo-Saxon ...	Münter	−0·6 (106)	0·0 (76)	+6·2 (66)	+1·2 (45)	−0·9 (43)	−0·3 (34)	+5·6 (30)	+1·7 (20)
French	Rollet	−1·0 (48)	+1·8 (45)	+4·2 (47)	+2·8 (43)	−0·2 (50)	+1·1 (46)	+5·1 (47)	+2·3 (42)
U.S.A. (White) ...	Hrdlička	−0·1 (200)	—	+2·4 (378)	+1·0 (182)	+0·1 (63)	—	+3·0 (133)	+3·0 (62)
Irish		−1·0 (22)	—	+2·4 (150)	+3·0 (109)	−0·3 (14)	—	+4·9 (137)	+3·0 (97)
German		−0·4 (86)	—	+3·1 (180)	+2·0 (93)	+1·0 (21)	—	+3·8 (45)	+4·0 (30)
Italian		+0·4 (39)	—	+2·1 (150)	+1·0 (53)	−2·0 (11)	—	+2·8 (122)	+2·0 (9)
Other White ...		−0·7 (53)	—	+2·7 (152)	+3·0 (89)	−2·0 (15)	—	+3·9 (48)	+1·0 (14)
Predynastic Egyptian	Warren	−1·3 (48)	−0·5 (53)	+3·6 (36)	+1·6 (16)	−1·5 (66)	−1·2 (88)	+5·6 (59)	+2·9 (36)
American Negro ...	Hrdlička	+1·9 (25)	—	+1·5 (56)	+1·4 (37)	−0·9 (13)	—	+4·0 (26)	+5·2 (17)
American Indian ...		−2·3 (99)	—	+1·4 (174)	+1·8 (81)	−1·3 (61)	—	+2·9 (127)	+0·9 (50)
Weighted Mean		−0·6 (726)	+0·3 (174)	+2·6 (1389)	+1·9 (748)	−0·7 (357)	−0·4 (168)	+3·7 (774)	+2·6 (377)

* The lengths used are the oblique of the femur, the maximum with spine of the tibia (except for the French series which gives the oblique length) and the maximum of the humerus and radius.

The individual bilateral differences (which can only be found, of course, for skeletons for which the lengths of both bones are available) were found, leading to

	Femur (106)	Tibia (76)	Humerus (66)	Radius (45)
Mean differences ...	$-0{\cdot}64 \pm {\cdot}25$	$-0{\cdot}02 \pm {\cdot}22$	$+6{\cdot}25 \pm {\cdot}33$	$+1{\cdot}16 \pm {\cdot}36$
Standard deviation of differences	$3{\cdot}82 \pm {\cdot}18$	$2{\cdot}90 \pm {\cdot}16$	$3{\cdot}99 \pm {\cdot}23$	$3{\cdot}57 \pm {\cdot}25$

There is a marked discordance between the means calculated in the two ways, and for samples of the sizes at present available it will clearly be unsafe to use any except those found from paired bones for the same skeletons. Several of the values in Table VIII are based on small numbers, and most of the series represent European races. There seems to be a remarkably close agreement between the various mean differences for the same sex and any particular bone and, assuming that the standard deviations given above for the male Anglo-Saxons can be applied to all the series, we may conclude that there is no evidence of racial differentiation in such constants. On this assumption, the weighted pooled means given may be considered, and it is probable that they all differ very significantly from zero except the two for the tibia. Excluding these, the common supposition that on the average the right arm bones are longer than the left, while the reverse is true for the femur, is amply confirmed. The bilateral differences are found to be decidedly greater for the arm than for the leg bones. There is no suggestion of a sex differentiation in these constants relating to the femur. Comparing the series singly, for nine of the ten the mean difference for the humerus is greater for females than males, but for the radius the female difference is the greater for six, there is equality in one case and the male is the greater for the remaining three. For the humerus, at least, there is clear indication that females are more asymmetrical than males. Two correlations between the bilateral differences for different bones were calculated in the case of the Anglo-Saxon male series, for the humerus and femur the value found is $-{\cdot}012 \pm {\cdot}096$ (49), and for the femur and tibia $+{\cdot}237 \pm {\cdot}084$ (58). The bilateral differences for different bones thus appear to be practically independent. The only living series giving the lengths of arms and legs on both sides is Thiele's. From his measurements we find that for 71 men aged 20 and over the mean length of the right arm exceeded that of the left by 3·6 mm., and the mean length of the *right* leg exceeded that of the left by 1·8 mm. The sign of this last difference is contrary to expectation.

Sex ratios may now be considered with reference to the comparative material. It has been demonstrated in the foregoing comparison that the difference between the means for the right and left sides shows a clear tendency to be greater for females than males in the case of the humerus, while there is some indication of the same sex differentiation for the radius, but none for the femur. Hence it will be expected that the sex ratio for the length of the humerus will tend to be greater for measurements of this bone on the left side than on the right side. In the case of all four bones considered the ratios were found, where possible, for the right and left sides separately, by using means based on all available

specimens, but excluding all which can only be given for fewer than forty. For the humerus the left ratios were found to be greater than the right in eight series out of nine—the exception being the Anglo-Saxon—and the maximum divergence is ·007. The side differences tend to be smaller for the other bones. In the case of these nine series the means were found of the paired ratios for the two sides separately, and then the averages of the pairs were taken, these being the values given in Table IX. For the other two series in the table—viz. the Swiss and the Eskimo—the mean lengths are only given for the right and left sides together, and these values were used to find the sex ratios.

Comparisons may be made first between the constants in Table IX for different series and a particular bone. The differences of this kind are all remarkably small, and there is no suggestion that the more "primitive" races (represented by the four lowest series in the table) tend to have either greater or lesser sex ratios

TABLE IX.

Sex Ratios for Lengths of the Long Bones (both sides together).*

Series	Author	Femur	Tibia	Humerus	Radius
Anglo-Saxon ...	Münter	1·086	1·086	1·077	1·106
Swiss (Alaman) ...	Schwerz	1·103	1·088	1·109	1·105
Norwegian	Wagner	1·106	1·109	1·081	1·080
U.S.A. (White) ...	Hrdlička	1·080	—	1·094	1·102
Four European ...	Hrdlička	1·076	—	1·093	1·100
French	Rollet	1·095	1·098	1·112	1·127
Lapp	Schreiner	1·074	1·088	1·082	1·094
Predynastic Egyptian	Warren	1·085	1·084	1·095	1·115
American Indian ...	Hrdlička	1·088	—	1·087	1·111
Pueblo Indian ...	Hooton	1·082	1·099	1·078	1·096
Western Eskimo ...	Hrdlička	1·089	1·081	1·081	1·117
Mean (unweighted)		1·086	1·092	1·091	1·105

* The lengths used are the oblique of the femur, the maximum including the spine of the tibia (except in the case of the French and Eskimo series for which the oblique length had to be used), and the maximum of the humerus and radius. The ways in which the ratios in this table were calculated are described in the text. All means involved for both sexes are based on 42 or more bones in the case of the separate series. The ratios given for the four European were found by first finding the average ratio of the two sides for each series, and then taking the mean of these four averages. All means for the separate series making up the four European are based on 32 or more bones.

than European races. The material does not permit ideal comparisons being made between the ratios for different bones, since it would be more correct, for this purpose, to base all the values for a particular series on the same set of complete skeletons, but some general conclusions can be derived from it as it stands. There is no clear suggestion of a significant difference between the ratios for the femur, tibia and humerus, but for eight of the eleven series the radius has the greatest value and this preponderance appears to be of significance.

The comparative material for lengths of the long bones is by no means extensive and confirmation of conclusions derived from it would be welcome. Unfortunately, there seem to be few data for corresponding male and female living series suitable for investigating the relationships discussed above. The Korean material is the best which was found. For six of the provinces both male and female means are given based on 55 or more individuals, and these give the following sex ratios:

	Leg Length *R**	Stature	Arm Length *R*†	Span
Range ...	1·072—1·096	1·076—1·099	1·085—1·105	1·089—1·106
Mean ...	1·083	1·087	1·095	1·097

For all but one of the six series the arm length and span show higher ratios than the leg length and stature, while the means for the latter pair are extraordinarily close to that for the femur in Table IX, and those for the former pair are between the means for the humerus and radius. In his essay on "Variation in Man and Woman" Professor Karl Pearson gave (pp. 294, 299 and 304) the means of three measurements taken on 1000 English men and 1000 English women. These lead to the sex ratios

Stature	Forearm *L*	Span
1·081	1·112	1·111

For the inhabitants of Fehmarn Island, Saller's means for 282—289 men and 318—324 women with ages between 25 and 60 give

Stature	Arm Length *R*‡	Span
1·070	1·089	1·094

The corresponding values for the different living series are remarkably close. Comparisons of the lengths of the arm bones suggested that the ratios for the left side are greater than those for the right side, and this should be remembered in comparing the English value for the forearm length with those for the total arm length in the case of the other two series. The ratios for measurements relating to arm lengths (including span) are greater for every series than those for leg length and stature, and this relation accords with the evidence of the long bones. There is also a close agreement between the ratios derived from the two kinds of material—comparing constants for arm measurements with those for arm bone measurements, and constants for leg measurements with those for leg bone measurements—and the data considered suggest forcibly that any racial differences there may be for the sex ratios considered are exceedingly small.

Comparisons may be made between the constants of variation for six male and six female skeletal series. For every one of these the vast majority of the male standard deviations are greater than the corresponding female values, and the longer bones have the larger constants. It has been shown above in the case of the Anglo-Saxon series that use of the coefficient of variation leads to a far

* Höhe des rechten vorderen oberen Darmbeinstachels: Martin No. 13.

† Ganze Armlänge rechts (Akromion-Daktylion): Martin No. 45.

‡ Höhe des rechten Akromion (M. 8)—Höhe der rechten Mittelfingerspitze (M. 11).

closer approach to equality in sexual comparisons, and in comparisons between the variabilities of different bones. It is also a better measure to use in comparing different series, and for these reasons it is the only one considered here. The coefficients of variation are given in Table X, and it should be realised that any particular one of these is based on all the bones available, so that any two constants for the same series may relate partly to the same skeletons, but also to two different sets of incomplete skeletons.

Sexual comparisons may be made first. There are 47 pairs of corresponding constants (omitting the mean values) and for 34 of these the male coefficient of variation is greater than the female, there is equality in one case and for the remaining 12 the female constant is the greater. The only differences which exceed 3·5 times their probable errors are for the length of the femur for the Lapps ($\Delta/(\text{p.e. }\Delta) = 3{\cdot}6$, the male value being the greater), the length of the right tibia for the Norwegians ($\Delta/(\text{p.e. }\Delta) = 3{\cdot}9$, the male value being the greater) and the length of the right radius for the Norwegians ($\Delta/(\text{p.e. }\Delta) = 3{\cdot}9$, the *female* value being the greater). It is suspected that the last relation is incorrect owing to an error in calculation or transcription*, and it may be noted that, considering all the series together, there is no clear suggestion of a peculiar preponderance of male over female variation, or the reverse, for any particular bone or for either side. There seems to be justification for the general conclusion that for all the bones male relative variation tends to be slightly greater than female, the difference being so small that it is only likely to be of statistical significance in the case of long series. This was the conclusion derived from the Anglo-Saxon material alone, but it must be examined further before being accepted. For five of the six series in Table X the numbers of male constants exceeding corresponding female constants are clearly greater than the numbers of female constants exceeding male constants, and for two of these (the Lapp and Pueblo Indian) every male value is greater than the corresponding female value. The series which is exceptional in this respect is the French, and it is the only one for which the sexes of the skeletons were known definitely. In this case there are six *female* constants greater than the male, there is equality in one case and only one length is found showing greater variability for the male series. Two possible explanations of this situation may be suggested, assuming in both cases that sexual differences in variability are actually the same for all the races represented. It may be supposed, in the first place, that inaccurate sexing of a certain number of skeletons in the case of all five series sexed anatomically has tended to give a fallacious preponderance of male over female relative variability, or, alternatively, the sex difference indicated by these may be correct, and some peculiar selection of the

* The Norwegian male coefficient of variation for the right radius is suspiciously small. For the male left radius and the female right and left radii the Norwegian constants are larger than any others (though the series does not show extreme variability in the case of the other bones), but for the male right radius the same series shows the smallest variation. For it the difference of the male coefficient for the right and left radii is far more significant than any other bilateral difference observed for any bone, and hence an error is suspected. The Norwegian male coefficient of variation for the right radius cannot be checked as individual measurements are not available.

TABLE X. *Coefficients of Variation for Lengths of the Long Bones*.*

Series	Male							
	Femur		Tibia		Humerus		Radius	
	R	*L*	*R*	*L*	*R*	*L*	*R*	*L*
Anglo-Saxon ...	5·03 ± ·19 (154)	5·10 ± ·21 (139)	5·81 ± ·27 (103)	6·34 ± ·28 (114)	5·55 ± ·24 (121)	4·86 ± ·21 (105)	5·88 ± ·32 (79)	5·48 ± ·29 (67)
Norwegian	4·60 ± ·14 (257)	4·54 ± ·13 (264)	5·19 ± ·15 (285)	5·15 ± ·15 (284)	4·77 ± ·18 (162)	5·01 ± ·19 (152)	4·57 ± ·23† (89)	6·02 ± ·30 (93)
French	5·43 ± ·35 (49)	5·07 ± ·35 (49)	4·89 ± ·33 (49)	4·84 ± ·34 (46)	4·66 ± ·31 (49)	4·94 ± ·34 (48)	4·80 ± ·32 (48)	5·10 ± ·36 (45)
Lapp	5·50 ± ·15 (305)		5·85 ± ·22 (156)	5·99 ± ·24 (140)	4·98 ± ·20 (148)	4·99 ± ·19 (150)	5·43 ± ·24 (120)	5·78 ± ·25 (118)
Predynastic Egyptian	5·47 ± ·29 (80)	5·24 ± ·28 (82)	4·94 ± ·26 (85)	5·70 ± ·30 (80)	5·22 ± ·32 (62)	5·39 ± ·29 (79)	5·02 ± ·35 (47)	4·45 ± ·31 (48)
Pueblo Indian ...	4·71 ± ·19 (145)	4·55 ± ·18 (142)	5·45 ± ·23 (130)	5·24 ± ·23 (117)	4·96 ± ·21 (128)	4·77 ± ·20 (134)	5·53 ± ·28 (91)	5·52 ± ·26 (100)
Mean (unweighted)	5·05	4·90	5·36	5·54	5·02	5·00	5·21	5·39
	Female							
Anglo-Saxon ...	5·33 ± ·34 (56)	5·18 ± ·33 (58)	5·59 ± ·40 (44)	5·06 ± ·35 (49)	5·02 ± ·35 (47)	4·39 ± ·31 (46)	5·36 ± ·44 (34)	4·96 ± ·41 (34)
Norwegian	4·57 ± ·14 (247)	4·29 ± ·13 (246)	4·41 ± ·13 (281)	4·68 ± ·14 (263)	5·65 ± ·22 (158)	5·37 ± ·21 (154)	6·11 ± ·32 (83)	5·84 ± ·30 (88)
French	5·43 ± ·37 (50)	5·29 ± ·36 (50)	5·57 ± ·38 (48)	5·48 ± ·37 (48)	5·12 ± ·35 (49)	5·47 ± ·38 (48)	5·10 ± ·34 (49)	4·66 ± ·34 (43)
Lapp	4·77 ± ·14 (259)		5·22 ± ·23 (119)	5·30 ± ·22 (130)	4·85 ± ·20 (131)	4·74 ± ·20 (132)	5·12 ± ·25 (97)	5·17 ± ·25 (97)
Predynastic Egyptian	4·93 ± ·22 (113)	4·52 ± ·21 (103)	4·90 ± ·22 (115)	4·93 ± ·21 (132)	5·01 ± ·24 (97)	4·86 ± ·21 (118)	4·58 ± ·27 (66)	5·11 ± ·29 (72)
Pueblo Indian ...	4·21 ± ·20 (106)	4·44 ± ·21 (104)	4·72 ± ·24 (90)	4·60 ± ·24 (81)	4·62 ± ·22 (96)	4·31 ± ·22 (87)	4·79 ± ·28 (68)	5·06 ± ·32 (57)
Mean (unweighted)	4·89	4·74	5·07	5·01	5·05	4·86	5·18	5·13

* The lengths used in this table are the same as in Table IX.

† Error suspected: see footnote to p. 278.

French material may have led to an exceptional preponderance of female over male variability in this particular case, so that no general conclusion can be deduced from it. New and better material will be needed to settle this question for long bones, but meanwhile we may conclude that the relative variabilities of their lengths are very closely similar for the two sexes; and this merely confirms the conclusion reached by Professor Karl Pearson in his essay written forty years ago.

Comparison between the variabilities found for the same bone on the right and left sides may be made next from the data in Table X. The Anglo-Saxon series alone suggested that the right arm bones are more variable than those of the left side, but the bilateral differences in variability for leg bones were less consistent. We are again unable to make exact comparisons since most of the pairs of corresponding constants for right and left sides in the table are not based entirely on the same skeletons. Neglecting the bilateral correlations in estimating the significance of the differences, only one out of the total 46 possible is found to exceed 3·5 times its probable error, viz. that for the radii of the male Norwegians for which the ratio of the difference to the square root of the sums of the squares of the two probable errors is 3·8. This coefficient of variation for the right side is suspected to be incorrect, however. Exact estimates of the significance of the bilateral differences (taking into account the bilateral correlations) would probably show that several are actually larger than any which could be attributed to chance. Crude comparisons may be made in another way. There is a total 46 of the kind considered, and for 27 the constants for the right side are the greater, while the position is reversed for the remaining 19. There is no case, for either sex, of all the right constants being greater than all the corresponding left for a particular bone. Taking both sexes together, for leg bones the coefficient for the right side is in excess in 14 cases and for the left side is in excess in eight cases; for arm bones the right side is in excess in 13 cases and the left in 11. Or, taking both sexes and the tibia and arm bones together, the figures are 19 *R* and 17 *L*: it is only for the femur alone, giving 8 *R* and 2 *L*, that there is really any suggestion of a bilateral difference in variability tending to be the same for all the series. The male unweighted mean coefficients given in the table do not confirm this, however, as they show a smaller side difference for the femur than for the tibia. The general conclusion that these differences are extremely small for all the bones—if they have real existence—seems to be fully justified. The suggestion derived from the Anglo-Saxon series alone is not confirmed.

A comparison of the coefficients of variation for different bones suggests the same sort of general conclusion. Here again correlations should be taken into account in estimating the significance of a particular difference, but a crude appreciation of the position can be obtained without their use. A comparison of the male and female constants separately leads to almost identical results and hence they are combined in the table below. This gives the number of cases in which one bone has a greater coefficient of variation than another, comparing right

side with right side and left side with left side, and then combining the totals for the two sides.

$F>T$ 6	$F>H$ 8	$F>R$ 8	$T>H$ 19	$T>R$ 13	$H>R$ 6
$T>F$ 14	$H>F$ 12	$R>F$ 12	$H>T$ 5	$R>T$ 11	$R>H$ 18

There is no bone which is more variable than any other in the case of all, or nearly all, the comparisons possible. Differences between the relative variabilities of different bones are extremely small and, on the supposition that they have real existence, the material here considered suggests that the tibia and radius tend to be relatively more variable than the femur and humerus, i.e. the lower bones of the limbs tend to be relatively more variable than the upper bones. It may be noted that the corresponding coefficients for different series in Table X are all remarkably close to one another, and there is no suggestion that the series representing more "primitive" races are either more or less variable than those representing civilised races.

There are few series for which it is possible to quote (or calculate) correlations between the lengths of different long bones, and the sizes of the samples available for this purpose are by no means large. Only one length of each bone is considered in treating the comparative material below, as it has been shown for the Anglo-Saxon series that two lengths of the same bone—such as the maximum and oblique of the femur—tend to have almost identical bilateral correlations. The bilateral correlations for three series are given in Table XI. All the differences between corresponding male and female values are quite insignificant, and for the eight pairs there are four cases in which the male correlation exceeds the female and four for which the reverse is true. Some very appreciable differences are found, however, between the corresponding correlations for different series. In every comparison of this kind (for both sexes) the Predynastic Egyptian series has the lowest correlation, and for all except one case (femur + tibia) the French has the highest. Only one significant difference is found between corresponding correlations for the Anglo-Saxon and French series (viz. for the tibia, the difference being 4·0 times its probable error), but there are several markedly significant differences between the values for the Egyptian series, on the one hand, and the Anglo-Saxon and French, on the other. This last relation is curious. Somewhat lower correlations of the kind considered might be expected for a peculiarly heterogeneous series, but the coefficients of variation (in Table X above) do not suggest that there is any appreciable difference in variability (and hence in racial homogeneousness) between the Egyptian and the two European series. Finally, it may be asked whether the data in Table XI provide any evidence of a difference in bilateral correlations between different bones. Comparisons of this kind must only be made between two values for the same series. There are 30 such possible and a difference greater than 3·5 times its probable error is only found in one case (viz. for the male French

TABLE XI.

Bilateral Correlations for Lengths of Long Bones.*

	Series	Male	Female
Femur	Anglo-Saxon ...	·9814 ± ·0024 (106)	—
	French	·9822 ± ·0034 (48)	·9863 ± ·0026 (50)
	Predynastic Egyptian	·9765 ± ·0045 (48)	·9618 ± ·0045 (66)
Tibia	Anglo-Saxon ...	·9811 ± ·0030 (69)	—
	French	·9941 ± ·0012† (45)	·9913 ± ·0017† (46)
	Predynastic Egyptian	·9634 ± ·0044 (63)	·9505 ± ·0047 (100)
Femur + Tibia	Anglo-Saxon ...	·9944 ± ·0010† (56)	—
	French	·9908 ± ·0019† (43)	—
Humerus	Anglo-Saxon ...	·9747 ± ·0042 (64)	—
	French	·9766 ± ·0046 (47)	·9855 ± ·0028 (47)
	Predynastic Egyptian	·9454 ± ·0091 (33)	·9643 ± ·0047 (52)
Radius	Anglo-Saxon ...	·9702 ± ·0059 (45)	—
	French	·9869 ± ·0027 (43)	·9807 ± ·0040 (42)
	Predynastic Egyptian	·9246 ± ·0157 (21)	·9322 ± ·0124 (27)
Humerus + Radius	French	·9835 ± ·0034 (42)	—

* For the Anglo-Saxon series the lengths used are the oblique of the femur, the "maximum including spine" of the tibia and the maximum of the humerus and radius. They are the same for the French series except in the case of the tibia which uses the "oblique" length. All these correlations were calculated by the writer. The Egyptian values are quoted from Warren (*loc. cit.*, Table XVII), the probable error of the male correlation for the femur being corrected: the lengths used here are the oblique of the femur, the length from "centre to centre of articular surfaces" of the tibia, the maximum of the humerus, and the length from "centre to centre" of the radius.

† Correlations found without grouping, all others in this and other tables being found by grouping.

tibia and humerus, the difference being 3·7 times its probable error), but no stress whatever can be laid on a single divergence indicating a low degree of significance when found in a set of comparisons. In spite of this lack of a clear distinction between different bones, there is a suggestion that the radius tends to give lower bilateral correlations than the other three bones, and that the femur and tibia tend to give higher values than the humerus, while the two leg bones appear to be quite undifferentiated from one another in this respect. A longer series will be needed to confirm these relationships.

It was felt that there would be some interest in finding the bilateral correlations for the two leg bones together and the two arm bones together, and three of this kind are given in the table. In the case of the Anglo-Saxons the correlation for the (femur + tibia) is actually quite significantly greater than those for either bone alone,

but it must be noted that the former correlation is based on fewer skeletons than the other two. In the case of the French series, for which the numbers are more alike, the (femur + tibia) correlation is between those for the femur and tibia alone, and the (humerus + radius) value is between those for the humerus and radius alone, while no significant differences are found. The conclusion derived from this is the best to accept at the moment.

Table XII gives the correlations, other than the bilateral, for the lengths of one leg bone with another and of one arm bone with another. The Chinese series, for

TABLE XII.

Correlations for Male Series between the Lengths of different Bones: Leg Bone with Leg Bone and Arm Bone with Arm Bone.*

	Series	Tibia *R*	Tibia *L*
Femur *R*	Anglo-Saxon ...	·8694 ± ·0172 (92)	·8649 ± ·0179 (90)
	French	·7822 ± ·0378 (48)	·7768 ± ·0399 (45)
	Predynastic Egyptian	·9331 ± ·0118 (55)	·9170 ± ·0153 (49)
	Chinese	·8904 ± ·0202 (48)	—
Femur *L*	Anglo-Saxon ...	·8631 ± ·0201 (73)	·8746 ± ·0159 (99)
	French	·8005 ± ·0350 (48)	·7951 ± ·0370 (45)
	Predynastic Egyptian	·9209 ± ·0142 (52)	·9490 ± ·0091 (56)
		Radius *R*	Radius *L*
Humerus *R*	Anglo-Saxon ...	·8969 ± ·0162 (66)	·8564 ± ·0252 (51)
	French	·8728 ± ·0234 (47)	·8593 ± ·0266 (44)
	Chinese	·6715 ± ·0534 (48)	—
Humerus *L*	Anglo-Saxon ...	·8866 ± ·0200 (52)	·8269 ± ·0299 (51)
	French	·8460 ± ·0283 (46)	·8564 ± ·0268 (45)

* The lengths used are the oblique of the femur, the "maximum including spine" of the tibia (except in the case of the French and Chinese series for which the oblique length had to be used) and the maximum of the humerus and radius.

which measurements of the bones of the right side only are given, can be included here, but the meagre female data available have been omitted. Comparisons between the different series may be made first. There are no significant differences between the corresponding Anglo-Saxon and French correlations, and in six cases the former are the greater while the position is reversed for the other two. In the case of the bilateral correlations the French constants tended to be greater than the Anglo-Saxon, but both sets of comparisons lead to the same conclusion in so far as they show that the difference between the two series in the respect considered is very slight. No such agreement is found from a comparison with the correlations for the

Egyptian series, however. The bilateral constants for this are less than those for the Anglo-Saxon and French in the case of all four bones and very significantly less in some instances, but the four Egyptian correlations in Table XII are all greater than the corresponding values for the two European series. The divergence of the Egyptian from the French correlation is 3·8 times its probable error in the case of the femur *R*—tibia *R*, and 4·0 times in the case of the femur *L*—tibia *L* correlation, and the divergence of the Egyptian from the Anglo-Saxon value is 4·1 times its probable error in the case of the femur *L*—tibia *L* comparison. This difference in the orders in which the series are arranged by the two classes of correlations is unexpected and curious. The Chinese constant relating to the two leg bones is not peculiar: its humerus *R*—radius *R* value is decidedly small and significantly less ($\Delta/(\text{p.e. } \Delta) = 4{\cdot}0$) than the Anglo-Saxon value. Further comparisons between the correlations in Table XII must be restricted, in general, to different pairs or groups relating to the series considered singly, owing to the observed racial peculiarities of these constants. There are eight Anglo-Saxon correlations and all the differences between these are quite insignificant; the same is true for the eight French and for the four Egyptian values, but the Chinese correlation for the leg bones exceeds that for the arm bones by 3·8 times the probable error of the difference. It follows from these relations that for any particular series (other than the imperfectly represented Chinese) there is unlikely to be any clear indication of differences between different combinations of the two sides, and for the Anglo-Saxon and French (Egyptian correlations for arm bones being omitted as the numbers for them are too small) there is unlikely to be any clear indication of suggestive differences between bones of the two limbs. Detailed comparisons confirm these points entirely, and, in particular, we must conclude (as from the Anglo-Saxon material alone, Table III) that there is no distinction between the correlations for bones *A* and *B*, belonging to the same member, on the same side and for bone *A* on one side and bone *B* on the other. Finally, comparisons may be made, series by series, between the correlations in Tables XI and XII. For the Anglo-Saxon and French, respectively, all the bilateral correlations are appreciably greater than all those of the other kind; the same is true, though the excess is much less, for the Egyptian correlations relating to leg bones only, but for this series there are some bilateral correlations for arm bones less than some of the correlations for leg bones in Table XII.

The third (and last) class of correlations to be considered is that relating to lengths of an arm bone with a leg bone: they are given in Table XIII. For every one of the 16 kinds of correlations shown there all the differences between the correlations for the different series are insignificant, and for each series considered separately the maximum differences between the different correlations are also insignificant. In spite of this uniformity, a few suggestions of distinction may be noted. In 11 cases the Anglo-Saxon correlation is in excess, and in the remaining five in defect, of the French. The same tendency was noted for the constants in Table XII, but the opposite tendency was found for the bilateral correlations (Table XI). The two European series are remarkably alike in the respect con-

TABLE XIII.

Correlations for Male Series between Lengths of different Bones: Leg Bone with Arm Bone.*

	Series	Femur *R*	Femur *L*	Tibia *R*	Tibia *L*
Humerus *R*	Anglo-Saxon ...	·8762 ± ·0157 (100)	·8665 ± ·0188 (80)	·8702 ± ·0193 (72)	·8405 ± ·0238 (69)
	French	·8275 ± ·0307 (48)	·8456 ± ·0276 (48)	·8522 ± ·0264 (49)	·8218 ± ·0326 (45)
	Chinese	·7377 ± ·0443 (48)	—	·6866 ± ·0514 (48)	—
Humerus *L*	Anglo-Saxon ...	·8432 ± ·0225 (75)	·8532 ± ·0209 (77)	·8062 ± ·0313 (57)	·8176 ± ·0277 (65)
	French	·8129 ± ·0334 (47)	·8544 ± ·0266 (47)	·8011 ± ·0352 (47)	·8262 ± ·0323 (44)
	Predynastic Egyptian	·9199 ± ·0168 (38)	·8271 ± ·0311 (47)	·8925 ± ·0200 (47)	·8413 ± ·0312 (40)
Radius *R*	Anglo-Saxon ...	·8590 ± ·0225 (62)	·7686 ± ·0366 (57)	·8065 ± ·0321 (54)	·8620 ± ·0240 (52)
	French	·7413 ± ·0443 (47)	·7643 ± ·0409 (47)	·8351 ± ·0298 (47)	·7662 ± ·0415 (45)
	Chinese	·6696 ± ·0536 (48)	—	·7642 ± ·0405 (48)	—
Radius *L*	Anglo-Saxon ...	·7812 ± ·0348 (57)	·7724 ± ·0360 (57)	·7924 ± ·0355 (50)	·8303 ± ·0280 (56)
	French	·7384 ± ·0462 (44)	·7826 ± ·0394 (44)	·7967 ± ·0371 (44)	·7798 ± ·0403 (43)

* The lengths used are the oblique of the femur, the "maximum including spine" of the tibia (except in the case of the French and Chinese series for which the oblique length had to be used) and the maximum of the humerus and radius.

sidered, however. It has been seen that the Predynastic Egyptian differs quite markedly from both in having lower bilateral correlations, but higher values of the arm bone—arm bone and leg bone—leg bone class. Three of the four Egyptian correlations in Table XIII are the highest of their kinds found. The four Chinese correlations are all the lowest of their kinds. Suggestions of differences associated with side, or different pairs of bones, are best appreciated from comparisons of the mean values given below.

	Anglo-Saxon	French	Egyptian	Chinese
R bone with *R* bone	·8530 (4)	·8140 (4)	—	·7145 (4)
L „ *L* „	·8184 (4)	·8107 (4)	·8342 (2)	—
R „ *L* „	·8201 (8)	·7934 (8)	·9062 (2)	—
Humerus and Femur	·8598 (4)	·8351 (4)	·8735 (2)	·7377 (1)
„ „ Tibia	·8336 (4)	·8253 (4)	·8669 (2)	·6866 (1)
Radius and Femur	·7953 (4)	·7567 (4)	—	·6696 (1)
„ „ Tibia	·8228 (4)	·7945 (4)	—	·7642 (1)
Total	·8278 (16)	·8029 (16)	·8702 (4)	·7145 (4)

The Anglo-Saxon series alone suggested that for the class considered the *R-R* correlations tend to be in excess of the *L-L* and *R-L* correlations. This relation

is not found for the French series, however, and it seems safest to conclude that in general the sides can be interchanged in all possible ways without affecting the degree of association found. The data available suggest, however, that for different pairs of bones the humerus-femur correlations tend to be the greatest and the radius-femur the smallest. For every series the correlation of the former kind exceeds the value corresponding to it (with regard to combinations of sides) of the latter. The humerus-tibia and radius-tibia correlations are not clearly distinguished from one another by their magnitudes.

Comparisons may be made between the correlations (for different series considered separately) in Table XIII and those in Tables XI and XII. The eight Anglo-Saxon values in Table XII range from ·8969 to ·8269, and the sixteen in Table XIII from ·8762 to ·7686. These ranges overlap but suggest that the correlations of the former group tend to be greater than those of the latter. The same is found for the other three series, and the tendency noted is made more evident by a comparison of mean values given below:

	Correlations (other than bilateral) between two bones of the same limb (Table XII)	Correlations between bones of different limbs (Table XIII)
Anglo-Saxon ...	·8673 (8)	·8278 (16)
French	·8236 (8)	·8029 (16)
Egyptian ...	·9300 (4)	·8702 (4)
Chinese	·7809 (2)	·7145 (4)

For every series all the bilateral correlations are greater than those in Table XIII. We may conclude in general, as for the Anglo-Saxon series considered alone, that bilateral correlations are higher than all others; those between two bones of different segments of the same limb are next in order and, while of the same order and still high, the correlations between bones of different limbs tend, on the average, to be the lowest of the classes considered. These general conclusions appear to be applicable to all races, and in formulating them it is not necessary to specify side or the positions of the bones (upper or lower) in a limb, since different categories for these two factors considered separately can apparently be interchanged without influencing the correlations to an appreciable extent.

Finally, we may consider the comparative material available for the indices derived from the lengths of different long bones. Owing chiefly to the fact that different definitions have been followed by different workers in determining the lengths, the two indices involving the tibia (and dealt with in Table VI above) can only be given for very few races, and they will be omitted here. Tables XIV and XV give collected data for the other two. The few standard deviations available for the radius-humerus index (Table XIV) give no clear indication of differences in these constants dependent on side or sex, and there are few significant differences between the corresponding values for different series. The

TABLE XIV.

Standard Deviations of Indices derived from Lengths of Long Bones.

Series	Author	$100 \times \frac{\text{Radius Max.}}{\text{Humerus Max.}}$			
		Male		Female	
		R	*L*	*R*	*L*
Anglo-Saxon ...	Münter	2·18 ± ·13 (62)	2·39 ± ·17 (47)	1·84 ± ·16 (31)	1·62 ± ·15 (26)
Swiss	Schwerz	2·62 ± ·16 (63)		3·23 ± ·25 (38)	
French ...	Rollet	1·82 ± ·13 (47)	2·00 ± ·14 (45)	2·06 ± ·14 (48)	2·35 ± ·17 (43)
Lapp	Schreiner	2·43 ± ·079 (216)		2·22 ± ·079 (179)	
Chinese ...	Stevenson	2·50 ± ·17 (48)	—	—	—
	Pooled*	2·35 (528)		2·29 (365)	
		$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$			
Anglo-Saxon ...	Münter	2·05 ± ·10 (100)	1·89 ± ·10 (77)	2·21 ± ·18 (36)	1·94 ± ·15 (36)
Swiss	Schwerz	1·95 ± ·12 (57)		2·06 ± ·14 (52)	
French ...	Rollet	2·05 ± ·14 (48)	2·04 ± ·14 (47)	1·93 ± ·13 (49)	1·91 ± ·13 (48)
Lapp	Schreiner	2·09 ± ·062 (257)		2·20 ± ·072 (215)	
Pueblo Indian	Hooton	2·38 ± ·11 (114)	2·37 ± ·10 (116)	1·98 ± ·10 (85)	2·14 ± ·12 (78)
	Pooled*	2·15 (816)		2·10 (599)	

* The pooled standard deviations were obtained by weighting the squares of the σ's given for the individual series with the numbers of skeletons on which they are based. The means in Table XV do not suggest that the sides are differentiated by the radius-humerus index, but the humerus-femur tends to be slightly greater on the right than on the left. The pooled standard deviations for the latter are hence slightly too large, in all probability, for either side considered separately, since some σ's for the two sides together were used in finding them.

pooled standard deviations for both sides together were obtained for each sex from the weighted squared σ's. The relations found for the humerus-femur index are the same, and the pooled values were found in the same way. Mean indices are given in Table XV for all the longer series available, some female values being omitted as the numbers for these are too small. Where possible the right and left sides are treated separately, but in several cases mean indices can only be given for the two sides together. Probable errors based on the same series as the means can only be given in a minority of cases, and in all others (marked with an asterisk) the pooled standard deviations were used. The few means in curled brackets were found not from individual values but from the means of the com-

TABLE XV. *Means of Indices derived from Lengths of Long Bones*.*

Series	Author	Side	$100 \times \frac{\text{Radius Max.}}{\text{Humerus Max.}}$		$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$	
			Male	Female	Male	Female
Anglo-Saxon ...	Münter	*R*	74·6±·19 (62)	73·5±·22 (31)	73·5±·14 (100)	73·6±·25 (36)
		L	75·4±·23 (48)	74·2±·21 (26)	72·2±·15 (77)	72·7±·22 (36)
Swiss	Schwerz	*R*+*L*	74·8±·22 (63)	73·3±·35 (38)	72·0±·17 (57)	72·1±·19 (52)
Norwegian	Wagner	*R*+*L*	{75·3±·12* (182)}	{75·1±·12* (172)}	{72·7±·09* (314)}	{74·4±·09* (312)}
U.S.A. (White) ...	Hrdlička	*R*	73·8±·12* (182)	72·8±·20* (62)	72·2±·11* (200)	71·8±·19* (63)
		L	73·9±·12* (182)	72·9±·20* (62)	71·7±·11* (200)	70·9±·19* (63)
Irish	,,	*R*	73·5±·15* (109)	72·8±·10* (97)	—	71·7±·26* (35)
		L	73·5±·15* (109)	73·1±·16* (97)	—	70·6±·26* (35)
German	,,	*R*	73·2±·16* (93)	72·0±·29* (30)	72·8±·17* (86)	—
		L	73·5±·16* (93)	71·8±·29* (30)	72·0±·17* (86)	—
Italian	,,	*R*	74·5±·22* (53)	—	72·5±·25* (39)	—
		L	74·6±·22* (53)	—	72·3±·25* (39)	—
White of other countries ...	,,	*R*	73·4±·17* (89)	—	73·9±·21* (53)	—
		L	73·3±·17* (89)	—	73·0±·21* (53)	—
French	Rollet	*R*	74·0±·18 (47)	72·2±·20 (48)	73·5±·20 (48)	72·3±·19 (48)
		L	74·0±·20 (45)	72·9±·24 (43)	72·4±·20 (47)	71·0±·19 (48)
Lapp	Schreiner	*R*+*L*	73·9±·11 (216)	73·3±·11 (179)	75·5±·09 (257)	74·3±·10 (215)
Predynastic Egyptian	Warren	*R*+*L*	79·3±·22* (52)	78·6±·17* (84)	70·9±·16* (80)	70·7±·13* (111)
American Indian ...	Hrdlička	*R*	77·8±·18* (81)	76·6±·22* (50)	72·4±·16* (99)	72·7±·20* (61)
		L	77·6±·18* (81)	76·8±·22* (50)	71·6±·16* (99)	71·8±·20* (61)
Pueblo Indian ...	Hooton	*R*	{77·2±·17* (91)}	{75·9±·19* (68)}	73·2±·15 (114)	73·8±·14 (85)
		L	{77·3±·16* (100)}	{76·0±·21* (57)}	72·7±·15 (116)	72·7±·16 (78)
Western Eskimo ...	Hrdlička	*R*+*L*	75·0±·14* (135)	73·1±·14* (133)	72·0±·10* (243)	71·8±·13* (153)
American Negro ...	,,	*R*+*L*	77·3±·18* (74)	77·2±·27* (34)	71·6±·22* (50)	70·2±·21* (52)
Chinese	Stevenson	*R*	76·6±·23 (48)	—	—	—

* The means in curled brackets were found from the means of the component lengths and all others are for indices of individual skeletons. The probable errors marked with an asterisk were found by using the pooled standard deviations given in Table XIV.

ponent lengths, and if the numbers for these are different, so that the two mean lengths are not based on precisely the same set of skeletons, the number given for the index is the smaller of the two. It is clear that means found in this way are likely to be particularly unreliable if the numbers involved are small, and that their probable errors are really larger than those given.

Considering the radius-humerus index first, nine comparisons can be made between a right and a left male mean for the same race: in five cases the left is greater than the right, in two the right is greater than the left and there is equality in two cases. For the seven female comparisons the left mean is greater than the right in six cases, and the reverse is true for the remaining one. But there is not a single difference of this kind, for either sex, which can be counted significant, and there is scarcely enough evidence to suggest that the index for the right side tends to be slightly less than that for the left. Sexual differentiation is perfectly clear, however, as every male index is greater than the corresponding female value, and some of the differences of this kind found are markedly significant. In making racial comparisons for the data in Table XV it is most convenient to use the average of the means for the two sides in cases where both are given for a particular race. In this way the following ranges of the index are found for the European series:

Male	*Female*
73·35—75·3 (10)	71·9—75·1 (8)

Markedly significant differences can be found between some of the means of the radius-humerus index for the different European series, but these are not arranged in any suggestive order. The Eskimo means fall within the European ranges, but all others in the table are decidedly greater than any available values for European series. The Chinese has an intermediate value and the Predynastic Egyptian is furthest removed from the European range*.

Turning to the humerus-femur index, there is evidence in this case of a distinct side difference (every right index being greater than every corresponding left and several of the differences being markedly significant), but no clear evidence of a sex difference which tends to be the same for all races. There are eight cases in which the male mean exceeds the female, seven for which the reverse is true and equality in one case. These numbers suggest that there may actually be equality in the means for the two sexes, but for five comparisons the male index is found to be significantly greater than the female, and there is only one female index significantly greater than the corresponding male value. The exception is for the Norwegian series, and the difference in this case is 13·4 times its probable error.

* Professor G. von Bonin has recently given collected radius-humerus indices for several American Indian series of long bones of which we have only used one (the Pecos Pueblo) as the numbers for the others are smaller than any considered here. ("The Skeletal material from the Lowry Area." *Publications of the Field Museum of Natural History, Anthropological Series*, Vol. XXIII., No. 1 (1936), p. 150.) The male means given for seven series range from 76·0 to 78·9 and for three female from 75·2 to 77·1. There is thus a complete separation between the ranges for European and American Indian series, if the Eskimo is omitted from the latter group.

The Norwegian series consists of isolated bones, and for males the length of the humerus is given for 314 and the length of the femur for 521 specimens; for the females the numbers are 312 and 493, respectively. The unusual sex difference may be due to the fact that the different means, in the case of either sex, are based on different sets of skeletons. Using, again, the average of the means for the two sides where both right and left are given, the ranges for the European series are:

Male	*Female*
71·95—75·5 (9)	71·15—74·4 (7)

Markedly significant differences may be found between the means for different European series, but the order in which they are arranged is not suggestive apart from the fact that an extreme value is found for the male Lapps. The non-European series fall within, or slightly below, the European ranges, and the racial differences here appear to be of far less account than in the case of the radius-humerus index.

5. *Summary and Conclusions.* Original individual measurements of the lengths of the long bones of the arms and legs belonging to 233 male and 93 female Anglo-Saxon adult skeletons are provided in the appended Tables XVI—XXI, and there are few skeletons for which all the lengths on both sides can be given. Means, constants of variability and correlations for these are presented and discussed in Section 2. Notes on previously published series of long bone measurements are provided in Section 3 together with those on a few living series which can be used in investigating the problems considered. This comparative material is less useful than it should be owing to the lack of standardisation of the techniques of measurement used, and to the fact that individual measurements are only available for a few of the series. The ulna and fibula are omitted in treating it. The following general conclusions, which are of both anthropological and anatomical interest, are reached:

(*a*) Professor Karl Pearson's formulae giving reconstructed statures from the lengths of different long bones, considered singly or in different combinations, lead to closely similar statures for the Anglo-Saxons, the best estimates being 1683 mm. for men and 1568 for women (Table VII). These averages are rather less than those for the general English population to-day, but they are greater than those for some classes of the community, and decidedly greater than the estimates available for seventeenth-century Londoners.

(*b*) The differences between the mean lengths of the bones for the right and left sides (Table VIII) are closely similar for different races (confining comparisons to a particular bone and the same sex), and there is no evidence of racial differentiation in this respect. The right arm bones are longer than the left, on the average, while the reverse is true for the femur, and there is insufficient evidence for the tibia to justify a general conclusion. The bilateral differences are decidedly greater for the arm than for the leg bones. There is no suggestion of a sex differentiation in these constants in the case of the femur, but for the

Division and County	Cemetery	Museum	Museum No.	Femora				Tibiae		Fibulae	
				Maximum		Oblique		Maximum		Maximum	
				R	*L*	*R*	*L*	*R*	*L*	*R*	*L*
A. Yorks.	Scarborough	R.C.S.	4·2901	468·0	—	466·0	—	384·0	—	366·0	—
,, ,,	,,	,,	4·2904	—	—	—	—	—	—	—	—
,, ,,	,,	,,	4·2911	440·5	—	438·5	—	365·0	—	—	—
,, ,,	,,	,,	4·2914	—	440·5	—	438·0	—	371·0	—	368·0
,, ,,	,,	,,	4·2915	—	479·0	—	475·0	—	382·0	—	367·0
,, ,,	,,	,,	4·2918	438·0	—	432·0	—	348·0	—	329·0	—
,, ,,	,,	,,	4·2919	430·0	—	428·0	—	371·0	—	350·0	—
,, ,,	,,	,,	4·2935	417·0	—	413·0	—	348·0	—	329·0	—
,, ,,	,,	,,	4·2942	465·0	—	459·0	—	363·0	—	358·0	—
,, Cambs.	Burwell	C.D.A.	12	454·8	455·7	451·8	452·0	382·9	381·0	368·8	367·8
,, ,,	,,	,,	27	—	456·5	—	453·3	373·5	—	—	—
,, ,,	,,	,,	31	446·0	447·8	444·5	446·0	—	345·0	—	—
,, ,,	,,	,,	34	455·5	458·0	454·4	457·4	357·6	354·4	—	—
,, ,,	,,	,,	56	474·2	478·7	472·0	476·0	391·8	390·0	—	—
,, ,,	,,	,,	58	446·4	452·0	440·8	448·5	—	352·1	—	343·5
,, ,,	,,	,,	64	491·5	495·0	490·0	493·1	399·1	400·0	—	—
,, ,,	,,	,,	66	443·6	443·0	433·7	439·0	363·6	—	—	—
,, ,,	,,	,,	68	461·9	460·0	459·6	458·8	363·4	362·5	—	—
,, ,,	,,	,,	70	463·8	472·5	459·1	467·0	394·5	—	—	—
,, ,,	,,	,,	77	500·2	497·3	495·7	496·0	—	438·0	—	—
,, ,,	,,	,,	78	—	446·7	437·2	444·5	373·5	375·6	—	—
,, ,,	,,	,,	93	448·2	447·6	444·0	445·1	374·0	377·5	—	—
,, ,,	,,	,,	121	470·5	472·3	465·9	464·0	388·0	388·4	—	—
,, ,,	,,	,,	123	469·1	467·4	467·7	464·2	390·2	389·6	—	—
,, ,,	,,	,,	124	481·4	483·6	479·9	480·9	399·5	401·0	—	—
,, ,,	,,	,,	128	—	—	—	—	—	—	—	—
,, ,,	,,	,,	132	465·9	461·6	458·3	459·0	—	375·0	—	—
,, ,,	Melbourn	,,	5	475·6	475·5	475·0	473·0	—	380·0	—	—
,, Suffolk	Mildenhall	,,	1	440·8	441·5	433·5	434·0	363·5	363·0	352·4	—
W.S. Oxon.	Chadlington	U.M.O.	E. 11·9 587	413·1	—	407·4	—	343·4	—	—	—
,, ,,	,,	,,	,, 597	456·0	—	448·5	—	349·5	348·5	336·5	334·4
,, ,,	,,	,,	,, 598	501·4	499·3	500·0	496·5	408·6	408·3	—	384·0
,, ,,	,,	,,	,, 604	441·0	446·1	434·4	439·4	351·0	355·6	—	—
,, ,,	,,	,,	,, 607	467·0	461·5	462·0	460·5	376·0	373·3	—	—
,, ,,	Yarnton	,,	,, 617	—	—	—	—	—	—	—	—
,, ,,	Hinton Down	,,	,, 642	480·4	480·9	476·9	477·6	389·7	388·2	—	—
,, Berks.	Abingdon	,,	,, 4	475·0	481·5	473·0	480·0	407·0	407·5	391·0	—
,, ,,	,,	,,	,, 11	489·0	494·0	486·0	488·0	426·0	429·0	—	412·5
,, ,,	,,	,,	,, 19	487·5	491·0	486·0	489·0	—	406·0	—	—
,, ,,	,,	,,	,, 23	—	453·0	—	451·5	—	362·5	—	—
,, ,,	,,	,,	,, 30	469·0	468·5	468·0	468·0	392·0	389·5	—	—
,, ,,	,,	,,	,, 35	—	—	—	—	—	—	—	—
,, ,,	,,	,,	,, 39	523·0	527·0	519·5	525·0	455·0	451·0	—	—
,, ,,	,,	,,	,, 44	457·6	461·0	450·5	457·0	384·5	380·0	353·0	356·0
,, ,,	,,	,,	,, 57	464·0	458·0	458·0	452·0	385·0	385·5	—	—
,, ,,	,,	,,	,, 58	452·0	455·0	449·0	451·0	370·0	372·5	360·0	361·0
,, ,,	,,	,,	,, 74	518·0	517·0	515·0	516·0	424·0	432·0	408·0	—
,, ,,	,,	,,	,, 80	435·0	437·0	429·0	431·0	372·0	376·0	361·0	364·5
,, ,,	,,	,,	,, 83	434·5	435·0	432·5	434·5	354·0	354·0	—	336·0
,, ,,	,,	,,	,, 90	462·3	467·5	457·5	460·2	367·5	368·6	364·1	365·6
,, ,,	,,	,,	,, 96	425·3	428·0	422·6	424·5	331·8	334·0	—	322·5
,, ,,	,,	,,	,, 98	486·5	479·5	481·8	476·5	397·0	396·1	—	—
,, ,,	,,	,,	,, 110	481·5	474·0	478·5	471·8	397·4	397·6	377·0	382·2
,, ,,	,,	,,	,, 111	459·6	465·1	455·6	460·5	372·8	378·0	—	—
,, ,,	,,	,,	,, 115	468·0	481·1	465·5	477·8	395·3	399·0	381·0	383·0
,, ,,	Wytham	,,	,, 509	438·8	439·0	436·0	436·0	373·1	373·1	—	—
,, ,,	Sutton Courtenay	,,	,, 610	478·5	480·0	477·0	479·0	392·0	—	—	—
,, ,,	Wytham	,,	,, 618	427·8	429·5	426·7	427·1	—	325·5	—	321·0
,, Wilts.	Winkelbury Hill	P.R.M.	8	—	460·0	—	—	368·0	364·0	—	—
,, ,,	,,	,,	14	474·0	472·0	—	—	366·0	370·0	—	—
,, ,,	,,	,,	16	481·0	487·0	—	—	397·0	396·0	—	—
,, ,,	,,	,,	18	429·0	434·0	—	—	336·0	337·0	—	—
,, ,,	,,	,,	27	486·0	491·0	—	—	405·0	405·0	—	—
,, ,,	Roche Court Down	R.C.S.	4·2532	464·0	464·0	460·0	460·0	377·0	377·0	372·0	372·0
S.S. Surrey	Coulsdon	,,	4·2685 *D*[1]	—	466·4	—	464·0	363·5	—	—	—
,, ,,	Guildown	,,	4·2747	470·0	475·0	467·0	470·0	—	387·0	—	—
J. Hants.	Meon Hill	,,	4·2540	464·0	470·0	463·5	469·5	389·0	—	—	369·0
,, ,,	,,	,,	4·2542	455·0	456·5	453·0	455·0	384·5	384·0	368·0	365·5
,, ,,	,,	,,	4·2543	488·0	487·5	485·0	485·0	412·0	409·0	—	395·0
,, ,,	,,	,,	4·2544	447·0	447·0	442·5	444·0	388·0	389·0	381·0	377·0

E XVI. *Individual Measurements of Male Anglo-Saxon Long Bones.*

	Fibulae		Humeri				Radii		Ulnae		100 × Radius Max. / Humerus Max.		100 × Humerus / Femur	
	Maximum		Maximum		Oblique		Maximum		Maximum					
	R	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	
	366·0	—	361·0	—	348·0	—	263·0	—	282·5	—	72·85	—	77·47	
	—	—	327·0	—	319·0	—	249·0	—	268·0	—	76·15	—	—	
	—	—	—	307·0	—	304·0	—	240·0	—	262·5	—	78·18	—	
	—	368·0	—	316·0	—	311·0	—	247·0	—	263·0	—	78·16	—	7
	—	367·0	—	336·5	—	329·0	—	252·5	—	276·0	—	75·94	—	7
	329·0	—	318·5	—	309·0	—	235·0	—	251·0	—	73·78	—	73·73	
	350·0	—	317·0	—	311·0	—	230·0	—	250·0	—	72·56	—	74·07	
	329·0	—	299·5	—	292·0	—	211·0	—	230·5	—	70·45	—	72·52	
	358·0	—	342·0	—	334·0	—	254·0	—	280·0	—	74·27	—	74·51	
	368·8	367·8	—	312·0	—	310·0	—	241·4	—	263·1	—	77·37	—	6
	—	—	338·0	—	335·2	—	256·5	257·1	—	280·8	75·89	—	—	
	—	—	335·8	323·8	327·2	315·4	242·9	252·8	270·1	265·0	72·33	78·07	75·55	7
	—	—	333·0	—	326·2	—	245·3	—	263·5	261·5	73·66	—	73·28	
	—	—	350·3	345·1	343·8	339·3	260·0	256·2	286·6	—	74·22	74·24	74·22	7
	—	343·5	325·9	314·2	318·7	309·0	231·1	—	257·9	—	70·91	—	73·98	7
	—	—	361·8	—	357·2	—	276·0	275·0	299·0	294·2	76·29	—	73·84	
	—	—	323·8	315·6	319·1	311·1	246·5	—	—	260·3	76·13	—	74·66	7
	—	—	327·2	—	320·2	—	234·4	231·4	261·5	255·7	71·64	—	71·19	
	—	—	324·2	320·5	312·2	310·6	—	256·3	—	274·5	—	79·97	70·62	6
	—	—	373·1	—	363·1	—	—	286·3	—	303·0	—	—	75·27	
	—	—	332·0	326·1	319·4	—	243·1	—	—	268·0	73·22	—	75·94	7
	—	—	320·0	—	312·1	—	246·0	248·0	264·8	266·8	76·88	—	72·02	
	—	—	—	336·9	—	334·1	254·5	250·0	—	270·7	—	74·21	—	7
	—	—	353·5	353·3	344·0	344·8	265·4	262·3	—	283·0	75·08	74·24	75·58	7
	—	—	359·8	352·0	347·0	343·3	—	264·5	—	285·0	—	75·14	74·97	7
	—	—	323·0	—	321·6	—	263·5	—	288·0	—	81·58	—	—	
	—	—	335·4	326·2	332·5	320·1	246·0	—	263·6	258·6	73·35	—	73·18	7
	—	—	—	351·9	—	343·8	—	255·5	270·4	—	—	72·61	—	7
	352·4	—	319·2	312·3	314·5	306·0	230·3	229·5	250·0	244·3	72·15	73·49	73·68	7
	—	—	306·0	304·3	303·4	302·3	—	240·9	271·4	—	—	79·17	75·11	
	336·5	334·4	349·6	350·5	345·0	349·3	—	248·0	268·6	—	—	70·76	77·59	
	—	384·0	363·3	358·0	358·0	356·8	263·5	264·8	286·0	287·3	72·47	73·97	72·66	7
	—	—	312·0	—	308·5	—	229·0	228·8	251·0	—	73·40	—	71·82	
	—	—	346·2	332·3	333·0	320·5	—	248·0	273·0	266·3	—	74·63	74·95	7
	—	—	363·0	—	353·4	—	264·0	—	284·5	—	72·73	—	—	
	—	—	348·0	342·7	—	336·4	256·5	252·8	—	276·7	73·71	73·77	72·97	7
	391·0	—	347·5	346·5	344·5	342·5	272·0	271·0	290·5	290·5	78·27	78·21	73·47	7
	—	412·5	369·5	360·0	361·0	352·0	277·5	275·0	302·0	301·0	75·10	76·39	76·03	7
	—	—	346·5	—	343·0	—	—	260·5	285·5	284·5	—	—	71·30	
	—	—	324·0	316·5	316·0	313·0	244·0	243·0	268·0	—	75·31	76·78	—	7
	—	—	350·0	341·5	341·0	337·0	256·0	248·5	274·0	271·0	73·14	72·77	74·79	7
	—	—	—	348·0	—	338·0	248·0	251·0	—	270·0	—	72·13	—	
	—	—	384·0	382·0	380·0	377·5	283·5	286·0	310·0	306·0	73·83	74·87	73·92	7
	353·0	356·0	337·0	329·0	329·5	321·5	251·5	249·5	276·0	274·0	74·63	75·86	74·81	7
	—	—	—	331·0	—	328·5	—	251·0	276·0	271·5	—	75·83	—	7
	360·0	361·0	342·0	343·0	337·5	337·0	256·5	258·0	271·0	273·0	75·00	75·22	76·17	7
	408·0	—	381·0	369·0	372·5	360·0	279·5	281·0	310·5	308·0	73·36	76·15	73·98	7
	361·0	364·5	319·0	315·0	316·0	308·0	239·0	236·0	—	258·5	74·92	74·92	74·36	7
	—	336·0	315·0	313·0	312·0	310·0	239·0	237·0	260·5	259·5	75·87	75·72	72·83	7
	364·1	365·6	—	333·8	—	328·0	242·1	229·1	263·8	257·2	—	68·63	—	7
	—	322·5	308·0	—	303·5	—	232·5	229·5	252·0	248·0	75·49	—	72·88	
	—	—	359·7	349·5	353·7	340·2	276·3	269·1	300·1	297·3	76·81	77·00	74·66	7
	377·0	382·2	349·3	335·6	342·1	329·8	251·5	255·0	271·1	—	72·00	75·98	73·00	7
	—	—	335·0	333·0	326·2	326·0	257·1	257·9	—	283·0	76·75	77·45	73·53	7
	381·0	383·0	362·0	347·7	352·6	339·7	274·0	274·4	292·8	287·8	75·69	78·92	77·77	7
	—	—	311·6	—	310·0	—	—	254·0	—	270·5	—	—	71·47	
	—	—	338·0	334·0	324·5	323·0	259·5	—	279·5	275·5	76·78	—	70·86	6
	—	321·0	317·1	305·5	313·8	303·2	226·0	—	242·0	—	71·27	—	74·31	7
	—	—	326·0	—	—	—	—	261·0	278·0	—	—	—	—	
	—	—	347·0	—	—	—	246·0	244·0	272·0	—	70·89	—	—	
	—	—	347·0	344·0	—	—	258·0	256·0	282·0	278·0	74·35	74·42	—	
	—	—	—	305·0	—	—	236·0	236·0	259·0	—	—	77·38	—	
	—	—	—	347·0	—	—	267·0	269·0	288·0	293·0	—	77·52	—	
	372·0	372·0	342·5	330·0	335·0	321·0	260·5	261·0	281·0	279·0	76·06	79·09	74·46	7
	—	—	325·0	—	310·0	—	244·0	—	264·0	—	75·08	—	—	
	—	—	329·5	331·0	323·0	327·0	—	256·0	—	278·0	—	77·34	70·56	7
	—	369·0	—	337·0	—	328·5	244·0	244·0	—	265·0	—	72·40	—	7
	368·0	365·5	—	327·5	—	323·5	242·0	239·0	269·0	—	—	72·98	—	7
	—	395·0	359·5	—	349·0	—	265·0	265·0	283·0	282·0	73·71	—	74·12	
	381·0	377·0	335·0	334·0	326·0	327·0	253·0	256·0	270·0	—	75·52	76·65	75·71	

$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{Tibia Obl.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{(Humerus Max. + Radius Max.)}}{\text{(Femur Obl. + Tibia Max.)}}$		Angle of Femur	
R	*L*	*R*	*L*	*R*	*L*	*R*	*L*
77·47	—	80·47	—	73·67	—	5°·3	—
—	—	—	—	—	—	—	—
—	—	80·50	—	—	—	6°·1	—
—	72·15	—	82·88	—	69·89	—	6°·1
—	70·84	—	78·42	—	69·23	—	7°·4
73·73	—	7[illegible]·84	—	71·47	—	9°·1	—
74·07	—	8[illegible]·93	—	68·46	—	5°·5	—
72·52	—	82·57	—	67·48	—	7°·9	—
74·51	—	76·91	—	72·42	—	9°·2	—
—	69·03	82·98	82·52	—	66·76	6°·5	7°·3
—	—	—	—	—	—	—	6°·8
75·55	72·60	—	75·78	—	73·17	4°·7	5°·1
73·28	—	76·94	75·73	71·57	—	4°·0	2°·9
74·22	72·50	81·31	80·25	70·98	69·76	5°·5	6°·1
73·98	70·06	—	76·72	—	—	9°·1	7°·1
73·84	—	79·82	79·50	72·16	—	4°·5	5°·1
74·66	71·89	81·99	—	71·89	—	12°·1	7°·7
71·19	—	77·33	77·27	68·57	—	5°·7	4°·1
70·62	68·63	84·19	—	—	—	8°·2	8°·8
75·27	—	—	86·69	—	—	7°·7	4°·1
75·94	73·36	83·60	82·70	71·29	—	—	5°·7
72·02	—	82·43	83·02	69·53	—	7°·9	6°·1
—	72·61	81·73	81·98	—	69·18	8°·0	10°·8
75·58	76·11	81·72	82·21	72·48	72·44	4°·4	6°·7
74·97	73·20	81·58	81·72	—	70·22	4°·5	6°·0
—	—	—	—	—	—	—	—
73·18	71·07	—	79·96	—	—	10°·4	6°·1
—	74·40	—	78·65	—	71·54	2°·9	5°·9
73·68	71·96	82·01	81·80	69·29	68·32	10°·4	10°·6
75·11	—	82·33	—	—	—	9°·5	—
77·59	—	76·16	—	—	—	10°·4	—
72·66	72·10	80·12	80·62	69·27	69·14	4°·3	6°·1
71·82	—	78·96	79·11	69·23	—	9°·9	9°·9
74·95	72·16	79·65	79·33	—	69·93	8°·4	3°·8
—	—	—	—	—	—	—	—
72·97	71·75	80·04	79·61	70·08	69·10	6°·9	6°·7
73·47	72·19	84·36	83·23	70·72	69·89	5°·3	4°·5
76·03	73·77	86·01	86·27	71·26	69·55	6°·4	8°·9
71·30	—	—	81·39	—	—	4°·5	5°·2
—	70·10	—	78·52	—	69·07	—	4°·7
74·79	72·97	82·05	81·52	71·13	69·13	3°·7	2°·7
—	—	—	—	—	—	—	—
73·92	72·76	86·04	84·38	68·78	68·72	6°·6	4°·9
74·81	71·99	83·57	81·40	70·82	69·45	9°·7	7°·6
—	73·23	82·31	83·52	—	69·83	9°·2	9°·3
76·17	76·05	80·62	80·82	73·44	73·34	6°·6	7°·6
73·98	71·51	80·78	82·17	70·64	68·86	6°·2	3°·5
74·36	73·09	84·85	85·38	70·01	68·62	9°·5	9°·5
72·83	72·04	80·00	79·63	70·80	70·11	6°·2	2°·8
—	72·53	78·58	78·36	—	68·25	8°·3	10°·2
72·88	—	76·62	76·80	72·03	—	6°·5	7°·3
74·66	73·35	80·74	81·45	72·70	71·22	7°·9	6°·4
73·00	71·13	81·39	82·58	68·91	68·25	6°·4	5°·5
73·53	72·31	80·07	80·35	71·82	70·81	7°·0	8°·1
77·77	72·77	83·20	81·83	74·23	71·28	5°·9	6°·7
71·47	—	83·74	83·74	—	—	6°·5	6°·7
70·86	69·73	80·50	—	69·08	—	4°·5	3°·7
74·31	71·53	—	74·34	—	—	4°·1	6°·1
—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—
74·46	71·74	80·00	80·43	72·22	70·78	7°·5	7°·5
—	—	—	—	—	—	—	5°·8
70·56	70·43	—	80·00	—	68·57	6°·5	8°·3
—	71·78	81·98	—	—	—	2°·7	2°·7
—	71·98	82·78	82·42	—	68·01	5°·4	4°·6
74·12	—	83·30	83·09	69·97	—	6°·4	5°·8
75·71	75·23	85·65	85·81	71·14	71·17	8°·1	6°·7

TABLE XVII.

Division and County	Cemetery	Museum	Museum No.	Femora				Tibiae	
				Maximum		Oblique		Maximum	
				R	*L*	*R*	*L*	*R*	*L*
A. Yorks.	Scarborough	R.C.S.	4·2906	—	—	—	—	—	347·0
,, ,,	,,	,,	4·2907	456·0	—	451·0	—	382·0	—
,, ,,	,,	,,	4·2917	423·5	432·0	420·5	426·0	353·0	356·0
,, ,,	,,	,,	4·2920	433·0	—	431·0	—	—	—
,, ,,	,,	,,	4·2922	476·0	—	471·0	—	383·0	—
,, ,,	,,	,,	4·2924	446·0	—	441·5	—	369·5	—
,, ,,	,,	,,	4·2928	—	437·0	—	433·0	—	356·0
,, ,,	,,	,,	4·2936	465·5	—	463·0	—	373·5	—
,, ,,	,,	,,	4·2938	450·0	—	446·0	—	370·0	—
,, ,,	,,	,,	4·2939	—	—	—	—	—	—
,, ,,	,,	,,	4·2947	423·0	—	418·0	—	—	—
,, ,,	,,	,,	4·2949	453·0	—	451·5	—	377·0	—
,, Cambs.	Burwell	C.D.A.	10	469·4	—	468·0	—	402·6	—
,, ,,	,,	,,	11	504·8	503·6	500·9	497·4	—	—
,, ,,	,,	,,	16	463·4	467·8	460·0	465·5	384·3	—
,, ,,	,,	,,	22	506·1	510·8	502·5	506·0	424·8	427·7
,, ,,	,,	,,	23	—	—	—	—	—	360·8
,, ,,	,,	,,	26	476·3	475·8	469·8	471·0	379·4	379·4
,, ,,	,,	,,	37	474·1	470·0	469·4	467·0	373·1	373·8
,, ,,	,,	,,	41	467·1	466·1	464·5	461·7	395·2	396·6
,, ,,	,,	,,	42	489·0	489·9	481·4	479·8	—	395·0
,, ,,	,,	,,	52	449·6	450·2	446·0	445·2	—	362·2
,, ,,	,,	,,	57	441·7	—	433·5	—	369·5	362·1
,, ,,	,,	,,	65	482·2	488·9	471·4	474·5	411·8	409·0
,, ,,	,,	,,	74	464·2	455·8	459·1	453·7	—	367·1
,, ,,	,,	,,	107	439·0	—	434·4	—	356·7	—
,, ,,	,,	,,	111	515·3	518·8	512·6	515·6	—	408·2
,, ,,	,,	,,	112	439·0	445·0	435·4	441·5	358·6	359·5
,, ,,	,,	,,	115	457·6	461·4	457·6	460·3	387·1	385·8
,, ,,	,,	,,	122	519·7	514·4	515·0	508·2	414·3	413·3
,, ,,	,,	,,	126	—	—	418·8	—	—	343·5
,, ,,	,,	,,	131	455·0	—	450·4	—	365·9	364·1
,, ,,	,,	,,	140	485·5	481·2	484·0	478·4	413·8	413·4
,, ,,	,,	,,	4473	465·2	461·9	464·7	461·1	364·9	—
,, ,,	Melbourn	,,	19	453·5	457·5	450·5	452·8	388·0	387·2
,, ,,	,,	,,	36a	442·5	440·0	441·3	439·3	—	369·1
,, ,,	,,	,,	41	—	—	—	—	—	—
,, ,,	,,	,,	*	—	—	—	—	—	—
,, Suffolk	Mildenhall	,,	2	—	—	—	—	—	—
,, ,,	,,	,,	5	430·2	431·4	429·5	430·4	349·7	348·3
W.S. Oxon.	Chadlington	U.M.O.	E. 11·9 599	437·0	433·4	433·6	431·5	—	357·0
,, Gloucester	Barrington	,,	,, 645	471·6	—	466·5	—	—	—
,, Berks.	Abingdon	,,	,, 43	—	425·0	—	424·5	358·0	361·0
,, ,,	,,	,,	,, 49	467·5	465·0	464·0	458·0	—	—
,, ,,	,,	,,	,, 62	467·0	467·0	—	—	—	—
,, ,,	,,	,,	,, 69	463·5	—	460·5	—	364·5	364·0
,, ,,	,,	,,	,, 97	—	—	—	—	—	—
,, ,,	Sutton Courtenay	,,	,, 603	—	—	—	—	—	—
,, ,,	Wytham	,,	,, 611	474·0	467·5	470·0	464·5	370·0	367·5
,, ,,	,,	,,	,, 631	497·0	499·1	493·1	496·0	—	—
,, ,,	Dorchester Dykes	,,	,, 635	—	—	—	—	—	439·0
,, ,,	East Shefford	R.C.S.	4·2441	435·0	434·0	429·0	427·0	349·5	—
,, ,,	,,	,,	4·2442	—	—	—	—	384·0	—
,, ,,	,,	,,	4·2466	—	—	—	—	—	—
,, Wilts.	Roche Court Down	,,	4·2524	469·0	470·0	468·0	468·0	—	383·5
,, ,,	Winkelbury Hill	P.R.M.	4	—	—	—	—	—	—
,, ,,	,,	,,	5	485·0	487·0	—	—	374·0	375·0
,, ,,	,,	,,	19	456·0	463·0	—	—	—	379·0
,, ,,	,,	,,	26	—	—	—	—	—	411·0
,, ,,	,,	,,	28[1]	—	448·0	—	—	—	357·0
,, ,,	,,	,,	28[2]	—	448·0	—	—	—	—
,, ,,	,,	,,	*I*	—	—	—	—	375·0	373·0
S.S. Surrey	Mitcham	R.C.S.	4·2647	—	481·0	—	479·0	—	386·0
,, ,,	,,	,,	4·2653	—	439·0	—	438·0	—	361·0
,, ,,	,,	,,	4·2656	435·0	437·0	434·0	436·0	351·5	354·5
,, ,,	,,	,,	4·2657	—	—	—	—	400·0	—
,, ,,	,,	,,	4·2685 *D*[2]	460·0	460·0	458·0	458·0	389·0	—
J. Hants.	Meon Hill	,,	4·2541	488·0	482·5	483·5	479·5	393·0	388·0
,, ,,	,,	,,	4·2545	—	457·5	—	457·0	—	389·0
,, ,,	,,	,,	4·2546	449·0	—	448·0	—	363·0	373·5
,, Isle of Wight	Ventnor	,,	4·26	430·0	428·0	428·0	426·5	—	350·0

BLE XVII. *Individual Measurements of Male Anglo-Saxon Long Bones* (cont.).

Tibiae		Fibulae		Humeri				Radii		Ulnae		100 × Radius Max / Humerus Ma	
Maximum		Maximum		Maximum		Oblique		Maximum		Maximum			
R	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*
—	347·0	—	329·0	—	316·0	—	307·0	—	228·0	—	251·0	—	72·15
382·0	—	377·0	—	331·0	—	325·0	—	253·0	—	—	—	76·44	—
353·0	356·0	—	—	316·3	—	299·5	—	—	—	256·5	—	—	—
—	—	—	—	300·0	—	293·5	—	238·0	—	—	—	79·33	—
383·0	—	—	—	—	—	—	—	269·0	—	289·0	—	—	—
369·5	—	345·0	—	335·0	—	330·0	—	—	—	—	—	—	—
—	356·0	—	349·0	—	310·0	—	306·0	—	—	—	—	—	—
373·5	—	353·5	—	323·0	—	—	—	—	—	—	—	—	—
370·0	—	—	—	326·0	—	323·5	—	—	—	274·0	—	—	—
—	—	—	—	—	329·0	—	321·0	—	234·0	—	—	—	71·12
—	—	—	—	315·0	—	306·0	—	235·0	—	—	—	74·60	—
377·0	—	371·5	—	334·5	332·5	324·0	323·0	251·0	—	—	—	75·04	—
402·5	—	—	—	348·9	—	342·6	—	—	—	297·0	—	—	—
—	—	—	—	358·2	355·6	348·6	347·1	278·5	—	301·2	—	77·75	—
384·3	—	—	—	348·3	344·8	342·9	340·5	—	—	—	284·6	—	—
424·8	427·7	—	404·8	368·0	—	360·5	—	—	—	—	303·5	—	—
—	360·8	—	—	—	—	—	—	—	—	—	268·1	—	—
379·4	379·4	—	—	338·0	—	329·2	—	254·3	254·4	—	—	75·24	—
373·1	373·8	—	—	—	—	—	—	—	242·2	—	269·2	—	—
395·2	396·6	378·3	—	337·9	339·0	326·0	329·4	251·3	251·0	—	—	74·37	74·04
—	395·0	—	—	349·6	342·9	334·2	331·8	261·6	—	—	—	74·83	—
—	362·2	—	—	334·0	—	330·8	—	229·0	—	—	—	68·56	—
369·6	362·1	358·4	358·4	322·3	317·8	—	308·2	—	—	—	—	—	—
411·8	409·0	—	—	—	—	—	—	269·0	—	—	—	—	—
—	367·1	—	—	321·6	—	—	—	—	235·5	—	—	—	—
356·7	—	—	—	335·8	332·0	334·8	326·1	—	254·6	—	—	—	76·69
—	408·2	—	—	—	—	—	—	—	—	298·7	—	—	—
358·6	359·5	—	—	316·4	—	312·0	—	233·6	235·0	—	—	73·83	—
387·1	385·8	369·3	365·9	—	—	—	—	—	252·9	—	274·0	—	—
414·3	413·3	—	—	366·4	366·3	357·1	354·0	—	—	—	298·4	—	—
—	343·5	—	—	—	302·6	—	301·2	231·5	—	—	—	—	—
365·9	364·1	—	—	—	—	—	—	249·0	245·4	267·2	263·0	—	—
413·8	413·4	—	—	—	—	—	—	—	—	303·0	—	—	—
364·9	—	—	—	332·5	325·7	327·0	321·0	249·6	—	—	—	75·07	—
388·0	387·2	—	—	344·3	340·1	333·0	330·5	—	—	270·4	—	—	—
—	369·1	—	—	336·8	—	330·8	—	—	—	270·9	—	—	—
—	—	—	—	—	316·8	—	312·6	—	—	262·0	—	—	—
—	—	—	—	—	354·8	—	—	273·0	—	—	—	—	—
—	—	—	388·3	—	—	—	—	243·5	—	262·3	—	—	—
349·7	348·3	—	—	—	—	—	—	—	237·6	—	259·8	—	—
—	357·0	—	—	316·7	310·8	315·3	310·0	248·8	—	—	—	78·56	—
—	—	—	—	—	—	—	—	—	—	290·6	—	—	—
358·0	361·0	—	—	—	—	—	—	250·5	248·5	—	—	—	—
—	—	—	—	—	340·0	—	334·0	—	257·0	—	—	—	75·59
—	—	—	—	353·5	347·0	—	—	263·0	—	—	—	74·40	—
364·5	364·0	—	—	321·0	316·5	313·5	310·0	—	248·0	—	—	—	78·36
—	—	—	—	—	—	—	—	233·0	—	255·0	250·6	—	—
—	—	—	—	358·0	352·0	348·0	342·0	—	271·8	—	—	—	77·22
370·0	367·5	366·0	361·0	—	—	—	—	—	—	—	273·0	—	—
—	—	—	—	369·8	362·0	353·5	350·9	271·0	270·9	—	—	73·28	74·83
—	439·0	420·3	—	—	—	—	—	276·6	—	—	—	—	—
349·5	—	—	342·0	328·5	317·5	320·5	315·0	—	—	—	—	—	—
384·0	—	377·5	—	350·0	345·0	340·5	338·0	257·0	—	—	—	73·43	—
—	—	319·0	312·0	—	—	—	—	—	—	—	—	—	—
—	383·5	—	365·0	342·5	334·5	329·5	323·0	—	254·5	—	—	—	76·08
—	—	—	—	323·0	—	—	—	255·0	—	—	—	78·95	—
374·0	375·0	—	—	350·0	346·0	—	—	251·0	—	—	—	71·71	—
—	379·0	—	—	341·0	336·0	—	—	—	—	272·0	271·0	—	—
—	411·0	—	—	—	—	—	—	—	—	293·0	—	—	—
—	357·0	—	—	—	—	—	—	245·0	—	263·0	—	—	—
—	—	—	—	316·0	—	—	—	—	—	—	259·0	—	—
375·0	373·0	—	—	—	—	—	—	—	—	—	264·0	—	—
—	386·0	—	—	341·5	335·0	—	331·0	250·0	—	—	—	73·41	—
—	361·0	—	—	—	328·0	—	323·0	255·0	—	—	—	—	—
351·5	354·5	—	—	—	316·0	—	306·0	239·4	—	—	—	—	—
400·0	—	387·0	—	—	356·0	—	—	—	—	—	—	—	—
389·0	—	—	—	—	—	—	—	—	—	—	278·0	—	—
393·0	388·0	390·0	—	—	337·0	—	329·0	252·0	—	—	—	—	—
—	389·0	—	—	316·5	—	314·0	—	241·0	230·0	—	—	76·15	—
363·0	373·5	—	359·5	321·0	317·0	318·0	313·0	—	—	—	252·5	—	—
—	350·0	—	—	318·0	—	310·0	—	—	—	264·0	263·0	—	—

...glo-Saxon Long Bones (cont.).

...lli	Ulnæ		100 × Radius Max. / Humerus Max.		100 × Humerus Max. / Femur Obl.		100 × Tibia Obl. / Femur Obl.		100 × (Humerus Max. + Radius Max.) / (Femur Obl. + Tibia Max.)		Angle of Femur	
...mum	Maximum											
L	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*
228·0	—	251·0	—	72·15	—	—	—	—	—	—	—	—
—	—	—	76·44	—	73·39	—	82·71	—	70·32	—	8°·5	—
—	256·5	—	—	—	75·27	—	81·69	81·69	—	—	6°·8	9°·6
—	—	—	79·33	—	69·61	—	—	—	—	—	5°·5	—
—	289·0	—	—	—	—	—	79·30	—	—	—	8°·3	—
—	—	—	—	—	75·88	—	82·11	—	—	—	8°·1	—
—	—	—	—	—	—	71·59	—	80·25	—	—	—	7°·7
—	—	—	—	—	69·76	—	[illegible]·05	—	—	—	5°·9	—
—	274·0	—	—	—	73·09	—	[illegible]·17	—	—	—	7°·7	—
234·0	—	—	—	71·12	—	—	—	—	—	—	—	—
—	—	—	74·60	—	75·36	—	—	—	—	—	8°·8	—
—	—	—	75·04	—	74·09	—	81·73	—	70·93	—	4°·6	—
—	297·0	—	—	—	74·55	—	84·32	—	—	—	4°·4	—
—	301·2	—	77·75	—	71·51	71·49	—	—	—	—	7°·1	9°·0
—	—	284·6	—	—	75·72	74·07	81·80	—	—	—	6°·9	5°·7
—	—	303·5	—	—	73·23	—	82·95	82·94	—	—	6°·8	7°·8
—	—	268·1	—	—	—	—	—	—	—	—	—	—
254·4	—	—	75·24	—	71·95	—	79·05	78·85	70·08	—	9°·5	8°·2
242·2	—	269·2	—	—	—	—	77·78	78·33	—	—	8°·1	6°·5
251·0	—	—	74·37	74·04	72·74	73·42	83·36	84·17	68·86	69·06	6°·1	7°·9
—	—	—	74·83	—	72·62	71·47	—	80·66	—	—	10°·1	11°·7
—	—	—	68·56	—	74·89	—	—	79·56	—	—	7°·3	8°·5
—	—	—	—	—	74·35	—	83·41	—	—	—	11°·1	—
—	—	—	—	—	—	—	85·66	84·51	—	—	12°·1	13°·9
235·5	—	—	—	—	70·05	—	—	79·15	—	—	8°·5	5°·5
254·6	—	—	—	76·69	77·30	—	80·27	—	—	—	8°·3	—
—	298·7	—	—	—	—	—	—	77·62	—	—	5°·9	6°·4
235·0	—	—	73·83	—	72·67	—	80·52	79·61	69·62	—	7°·4	7°·2
252·9	—	274·0	—	—	—	—	82·85	82·08	—	—	0°·0	4°·0
—	—	298·4	—	—	71·22	72·08	78·89	79·75	—	—	7°·7	8°·9
—	—	—	—	—	—	—	—	—	—	—	—	—
245·4	267·2	263·0	—	—	—	—	79·46	—	—	—	8°·1	—
—	303·0	—	—	—	—	—	83·84	84·74	—	—	4°·5	6°·2
—	—	—	75·07	—	71·55	70·64	76·80	—	70·56	—	2°·7	3°·3
—	270·4	—	—	—	76·43	75·11	84·35	83·75	—	—	6°·6	8°·2
—	270·9	—	—	—	76·32	—	—	82·20	—	—	4°·2	3°·2
—	262·0	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—
—	262·3	—	—	—	—	—	—	—	—	—	—	—
237·6	—	259·8	—	—	—	—	79·56	79·07	—	—	3°·2	3°·9
—	—	—	78·56	—	73·04	72·03	—	80·80	—	—	7°·1	5°·4
—	290·6	—	—	—	—	—	—	—	—	—	8°·4	—
248·3	—	—	—	—	—	—	—	83·16	—	—	—	2°·8
257·0	—	—	—	75·59	—	74·24	—	—	—	—	7°·0	6°·9
—	—	—	74·40	—	—	—	—	—	—	—	—	—
248·0	—	—	—	78·36	69·71	—	77·42	—	—	—	6°·5	—
—	255·0	250·6	—	—	—	—	—	—	—	—	—	—
271·8	—	—	—	77·22	—	—	—	—	—	—	—	—
—	—	273·0	—	—	—	—	77·02	77·40	—	—	7°·5	6°·5
270·9	—	—	73·28	74·83	74·99	72·98	—	—	—	—	7°·2	6°·4
—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	76·57	74·36	79·72	—	—	—	9°·5	10°·3
—	—	—	73·43	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—
254·5	—	—	—	76·08	73·18	71·47	—	80·13	—	69·62	3°·7	5°·3
—	—	—	78·95	—	—	—	—	—	—	—	—	—
—	—	—	71·71	—	—	—	—	—	—	—	—	—
—	272·0	271·0	—	—	—	—	—	—	—	—	—	—
—	293·0	—	—	—	—	—	—	—	—	—	—	—
—	263·0	—	—	—	—	—	—	—	—	—	—	—
—	—	259·0	—	—	—	—	—	—	—	—	—	—
—	—	264·0	—	—	—	—	—	—	—	—	—	—
—	—	—	73·21	—	—	69·94	—	79·54	—	—	—	5°·2
—	—	—	—	—	—	74·89	—	80·82	—	—	—	3°·9
—	—	—	—	—	—	72·48	79·26	79·36	—	—	3°·9	3°·9
—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	278·0	—	—	—	—	82·97	—	—	—	5°·4	5°·4
—	—	—	—	—	—	70·28	79·52	79·46	—	—	7°·8	6°·4
230·0	—	—	76·15	—	—	—	—	84·03	—	—	—	2°·7
—	—	252·5	—	—	71·65	—	79·24	—	—	—	3°·8	—
—	264·0	263·0	—	—	74·38	—	—	80·42	—	—	5°·5	4°·8

TABLE XVIII. *Individual Measurements of Male Anglo-Saxon Fe*

Division and County	Cemetery	Museum	Museum No.	Femora				Tibiae		Hum	
				Maximum		Oblique		Maximum		Maximum	
				R	*L*	*R*	*L*	*R*	*L*	*R*	*L*
A. Yorks.	Scarborough	R.C.S.	4·2912	404·0	—	400·5	—	335·5	—	312·0	—
„ Cambs.	Burwell	C.D.A.	15	448·5	444·5	445·4	442·7	353·9	349·9	328·5	320·2
„ „	„	„	24	—	471·4	—	468·5	—	—	—	324·5
„ „	„	„	28	494·0	487·6	492·5	485·2	—	411·0	—	—
„ „	„	„	43	477·8	478·2	476·5	475·6	382·3	—	354·2	—
„ „	„	„	44	—	422·5	—	419·2	—	—	—	325·3
„ „	„	„	47	455·0	450·6	451·8	448·2	358·2	354·2	—	335·0
„ „	„	„	67	481·3	472·2	480·0	471·3	396·0	397·4	—	—
„ „	„	„	89	—	457·1	—	452·9	—	399·0	365·1	—
„ „	„	„	105	444·0	437·0	436·0	432·1	—	353·0	—	—
„ „	„	„	108	463·8	464·5	459·4	458·4	—	354·2	—	321·4
„ „	„	„	110	—	—	—	—	334·0	—	—	—
„ „	„	„	127	438·0	436·0	426·3	426·4	—	—	325·6	—
„ „	„	„	133	440·6	—	440·6	—	—	—	—	339·8
„ „	Melbourn	„	4	465·0	465·0	462·2	458·8	371·6	372·3	329·2	—
„ „	„	„	31	—	462·8	—	456·7	—	385·3	—	—
„ „	„	„	36	—	439·4	—	438·1	—	—	326·0	—
„ „	„	„	45	—	457·0	—	455·0	381·4	381·0	—	316·5
„ „	Haslingfield	U.M.O.	E. 11·9 482	453·0	—	448·9	—	365·0	367·0	—	—
E.S. Essex	Bartlow Hills	R.C.S.	4·2	—	508·0	—	507·0	—	—	—	—
W.S. Oxon.	Cassington	U.M.O.	E. 11·9 585	—	—	—	—	427·5	425·0	400·5	384·0
„ „	Chadlington	„	„ 591	—	427·1	—	426·2	360·1	355·6	—	—
„ „	„	„	„ 595	—	430·2	—	—	—	—	—	338·9
„ „	Yarnton	„	„ 29·10 76	—	428·8	—	424·5	—	358·5	—	309·5
„ Berks.	Abingdon	„	„ 33	455·5	449·0	451·0	446·0	—	364·0	—	—
„ „	„	„	„ 41	499·5	501·0	496·0	498·0	—	413·0	338·0	331·0
„ „	„	„	„ 47	440·0	439·0	437·0	437·0	—	368·0	331·0	325·0
„ „	„	„	„ 48	480·5	479·0	476·5	477·5	—	—	336·0	—
„ „	„	„	„ 52	482·5	482·5	477·5	478·5	407·5	405·5	348·0	340·0
„ „	„	„	„ 89	—	493·0	—	489·0	—	396·0	—	337·5
„ „	„	„	„ 107	469·6	469·0	465·0	466·7	—	—	—	333·0
„ „	Long Wittenham	„	„ 360	504·0	507·1	502·0	505·0	—	437·7	—	—
„ „	Wytham	„	„ 615	—	—	—	—	—	367·8	310·6	—
„ „	Frilford	„	„ 20·5 74	—	513·4	—	507·3	—	419·4	—	—
„ Wilts.	Alvediston	R.C.S.	4·2495	493·0	—	488·0	—	—	—	—	355·5
„ „	Roche Court Down	„	4·2520	—	551·5	—	546·5	—	427·0	—	—
„ „	„	„	4·2523	451·0	455·0	446·0	451·0	—	374·0	320·0	—
„ „	„	„	4·2529	467·0	465·0	464·5	459·0	—	—	345·0	336·5
„ „	„	„	4·2530	—	448·0	—	446·5	—	—	342·0	334·0
„ „	„	„	4·2531	491·5	493·5	483·5	487·5	400·0	400·0	353·0	343·0
„ „	„	„	4·2534	471·0	—	462·0	—	374·5	—	—	—
„ „	„	„	4·2980	446·0	—	445·0	—	—	—	—	314·0
„ „	„	„	4·2981	431·0	—	430·0	—	—	—	315·0	—
„ „	„	„	4·2982	—	454·0	—	452·0	—	—	—	321·5
„ „	„	„	4·2983	—	503·0	—	500·0	—	—	368·0	—
„ „	„	„	4·2984	466·0	—	462·0	—	—	—	338·0	—
„ „	„	„	4·2985	—	438·5	—	438·5	—	—	—	317·0
„ „	„	„	4·2986	442·0	—	436·0	—	—	—	303·0	—
„ „	„	„	4·2987	506·0	—	504·0	—	—	—	343·0	—
„ „	„	„	4·2988	—	482·0	—	479·5	—	—	—	336·0
„ „	„	„	4·2989	475·0	—	472·5	—	—	—	338·0	—
„ „	„	„	4·2990	466·0	—	461·0	—	—	—	331·0	—
„ „	„	„	4·2992	455·0	—	452·0	—	—	—	329·0	—
„ „	„	„	4·2994	447·0	—	442·0	—	—	—	321·0	—
„ „	„	„	4·2995	454·0	—	449·0	—	—	—	—	332·0
„ „	Winkelbury Hill	P.R.M.	13	464·0	462·0	—	—	372·0	—	336·0	330·0
S.S. Surrey	Mitcham	R.C.S.	4·2649	—	—	—	—	—	412·0	—	—
„ „	„	„	4·2650	463·0	466·0	461·0	462·5	381·5	—	345·0	337·0
„ „	„	„	4·2654	—	—	—	—	392·0	—	—	—
„ „	Coulsdon	„	4·2685 *B*	480·0	481·5	474·0	474·5	—	380·5	—	—
„ „	„	„	4·2685 *G*	436·0	—	435·0	—	347·0	—	—	—
J. Hants.	Meon Hill	„	4·2541*	—	—	—	—	—	390·5	—	—
„ „	„	„	4·2548	—	—	—	—	378·0	378·5	—	—
„ „	„	„	4·2549	446·0	451·0	441·5	443·5	389·0	389·0	—	—

...le Anglo-Saxon Femora, Tibiae and Humeri.

...ibiae		Humeri				$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{Tibia Obl.}}{\text{Femur Obl.}}$		Angle of Femur	
...ximum		Maximum		Oblique							
	L	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*
	—	312·0	—	307·5	—	77·90	—	81·77	—	7°·5	—
	349·9	328·5	320·2	321·0	315·0	73·75	72·33	77·66	77·23	6°·8	5°·2
	—	—	324·5	—	—	—	69·26	—	—	—	6°·4
	411·0	—	—	—	—	—	—	—	83·06	4°·4	5°·7
	—	354·2	—	340·6	—	74·33	—	78·55	—	4°·2	6°·0
	—	—	325·3	—	316·0	—	77·60	—	—	—	7°·2
	354·2	—	335·0	—	329·2	—	74·74	77·51	77·24	6°·8	6°·0
	397·4	—	—	—	—	—	—	80·83	82·62	4°·2	3°·5
	399·0	365·1	—	356·1	—	—	—	—	86·33	—	7°·8
	353·0	—	—	—	—	—	—	—	79·84	10°·9	8°·6
	354·2	—	321·4	—	317·0	—	70·11	—	75·52	7°·9	9°·3
	—	—	—	—	—	—	—	—	—	—	—
	—	325·6	—	321·3	—	76·38	—	—	—	13°·3	12°·0
	—	—	339·8	—	335·8	—	—	—	—	0°·0	—
	372·3	329·2	—	325·5	—	71·22	—	78·67	79·40	6°·3	9°·4
	385·3	—	—	—	—	—	—	—	84·37	—	9°·3
	—	326·0	—	317·4	—	—	—	—	—	—	4°·4
	381·0	—	316·5	—	312·6	—	69·56	—	81·97	—	5°·4
	367·0	—	—	—	—	—	—	79·53	—	7°·7	—
	—	—	—	—	—	—	—	—	—	—	3°·4
	425·0	400·5	384·0	387·4	372·0	—	—	—	—	—	—
	355·6	—	—	—	—	—	—	—	81·56	—	3°·7
	—	—	338·9	—	330·0	—	—	—	—	—	—
	358·5	—	309·5	—	306·8	—	72·91	—	82·57	—	8°·1
	364·0	—	—	—	—	—	—	—	79·82	8°·1	6°·6
	413·0	338·0	331·0	333·0	325·0	68·15	66·47	—	81·33	6°·8	6°·3
	368·0	331·0	325·0	316·0	316·0	75·74	74·37	—	82·38	6°·7	5°·5
	—	336·0	—	333·0	—	70·51	—	—	—	7°·4	4°·5
	405·5	348·0	340·0	345·0	337·0	72·88	71·06	83·66	83·07	8°·3	6°·9
	396·0	—	337·5	—	335·0	—	69·02	—	79·35	—	7°·3
	—	—	333·0	—	327·5	—	71·35	—	—	8°·0	5°·7
	437·7	—	—	—	—	—	—	—	85·09	5°·1	5°·3
	367·8	310·6	—	306·3	—	—	—	—	—	—	—
	419·4	—	—	—	—	—	—	—	81·10	—	8°·8
	—	—	355·5	—	346·5	—	—	—	—	8°·1	—
	427·0	—	—	—	—	—	—	—	76·85	—	7°·7
	374·0	320·0	—	310·0	—	71·75	—	—	80·60	8°·5	7°·6
	—	345·0	336·5	337·0	331·0	74·27	73·31	—	—	5°·9	9°·2
	—	342·0	334·0	336·0	329·0	—	74·80	—	—	—	4°·7
	400·0	353·0	343·0	343·0	327·0	73·01	70·36	80·35	79·90	10°·3	8°·9
	—	—	—	—	—	—	—	73·91	—	11°·1	—
	—	—	314·0	—	310·5	—	—	—	—	3°·8	—
	—	315·0	—	310·5	—	73·26	—	—	—	3°·9	—
	—	—	321·5	—	317·0	—	71·13	—	—	—	5°·4
	—	368·0	—	361·5	—	—	—	—	—	—	6°·3
	—	338·0	—	329·0	—	73·16	—	—	—	7°·5	—
	—	—	317·0	—	310·0	—	72·29	—	—	—	0°·0
	—	303·0	—	298·0	—	69·50	—	—	—	9°·5	—
	—	343·0	—	333·0	—	68·06	—	—	—	5°·1	—
	—	—	336·0	—	327·0	—	70·07	—	—	—	6°·4
	—	338·0	—	334·0	—	71·53	—	—	—	5°·9	—
	—	331·0	—	324·0	—	71·80	—	—	—	8°·4	—
	—	329·0	—	325·0	—	72·79	—	—	—	6°·6	—
	—	321·0	—	316·0	—	72·62	—	—	—	8°·6	—
	—	—	332·0	—	328·0	—	—	—	—	8°·5	—
	—	336·0	330·0	—	—	—	—	—	—	—	—
	412·0	—	—	—	—	—	—	—	—	—	—
	—	345·0	337·0	336·0	327·0	74·84	72·86	81·24	—	5°·3	7°·0
	—	—	—	—	—	—	—	—	—	—	—
	380·5	—	—	—	—	—	—	—	78·50	9°·1	9°·8
	—	—	—	—	—	—	—	78·39	—	3°·9	—
	390·5	—	—	—	—	—	—	—	—	—	—
	378·5	—	—	—	—	—	—	—	—	—	—
	389·0	—	—	—	—	—	—	86·30	85·91	8°·1	10°·5

TABLE XIX. *Individual Measurements of Male and Female Anglo-Saxon Femora and Humeri.*

Division and County	Cemetery	Museum	Museum No.	Male Femora					
				Maximum		Oblique		Angle of Femur	
				R	*L*	*R*	*L*	*R*	*L*
A. Cambs.	Haslingfield	U.M.O.	E. 11·9 477	454·0	452·5	446·0	446·0	10°·8	9°·7
,, ,,	,,	,,	,, 486	470·0	477·6	468·1	473·3	5°·2	7°·7
,, ,,	Burwell	C.D.A.	X 2	506·2	—	503·9	—	5°·5	—
,, ,,	,,	,,	45	457·6	—	450·0	—	10°·3	—
,, ,,	Melbourn	,,	22	466·1	471·0	465·1	469·0	3°·8	5°·3
,, ,,	,,	,,	28	—	451·4	—	450·3	—	4°·0
,, ,,	,,	,,	29	—	451·6	—	450·0	—	4°·9
,, Suffolk	Mildenhall	,,	60	—	—	—	482·2	—	—
W.S. Oxon.	Yarnton	U.M.O.	8 Feb. 77	478·7	480·3	473·8	475·0	8°·2	8°·5
,, Berks.	Abingdon	,,	E. 11·9 2	—	431·0	—	430·0	—	3°·5
,, ,,	Wytham	,,	,, 510	476·0	475·4	471·1	469·0	8°·2	9°·4
,, ,,	,,	,,	6 June 70	456·0	—	451·5	—	8°·1	—
,, ,,	Frilford	,,	30 Mar. 74	466·7	—	466·6	—	1°·2	—
,, ,,	,,	,,	June 74	—	471·0	—	468·0	—	6°·5
,, Wilts.	Winkelbury Hill	P.R.M.	31[2]	466·0	—	—	—	—	—
,, ,,	West Chisenbury	R.C.S.	4·2490	478·0	—	475·0	—	6°·4	—
S.S. Surrey	Mitcham	,,	4·2665	484·0	490·0	483·5	489·0	2°·6	3°·6
,, ,,	Coulsdon	,,	4·2685 *F*	—	469·0	—	466·0	—	6°·3
,, ,,	Banstead	,,	4·269	447·0	—	441·0	—	9°·4	—
,, Sussex	Southwick Down	U.M.O.	E. 11·9 622	488·1	—	486·8	—	4°·2	—
J. Hants.	Droxford	R.C.S.	4·255	481·0	—	475·0	—	9°·0	—
				Female Femora					
A. Cambs.	Burwell	C.D.A.	39	424·6	425·4	416·1	416·8	11°·5	11°·5
W.S. Oxon.	Yarnton	U.M.O.	E. 11·9 506	416·0	422·5	414·5	419·5	4°·8	6°·8
,, Berks.	Abingdon	,,	,, 76	—	409·5	—	404·0	—	9°·4
,, ,,	,,	,,	,, 95	—	423·0	—	420·0	—	6°·8
,, ,,	,,	,,	,, 100	369·6	—	367·5	—	6°·1	—
,, ,,	East Shefford	R.C.S.	4·2456	—	438·0	—	—	—	—
				Male Humeri					
A. Cambs.	Burwell	C.D.A.	92	354·1	—	350·5	—		
W.S. Berks.	Abingdon	U.M.O.	E. 11·9 55	327·0	322·0	317·0	309·5		
,, ,,	Wytham	,,	,, 643	—	322·5	—	317·5		
,, ,,	East Shefford	R.C.S.	4·2444	—	321·0	—	318·0		
,, Oxon.	Chaddington	U.M.O.	E. 11·9 596	—	331·0	—	324·0		
,, Wilts.	Roche Court Down	R.C.S.	4·2991	327·0	—	320·0	—		
S.S. Surrey	Mitcham	,,	4·2652	337·0	—	330·0	—		
				Female Humeri					
A. Cambs.	Burwell	C.D.A.	97	—	325·0	—	320·2		
,, Yorks.	Scarborough	R.C.S.	4·2903	—	—	—	283·0		
W.S. Wilts.	Winkelbury Hill	P.R.M.	3	268·0	260·0	—	—		
,, ,,	,,	,,	17	315·0	313·0	—	—		

UNIV. OF MICH.

TABLE XX. *Individual*

Division and County	Cemetery	Museum	Museum No.	Femora				Tibiae		Fibulae		
				Maximum		Oblique		Maximum		Maximum		Maxim
				R	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*
A. Yorks.	Scarborough	R.C.S.	4·2909	—	425·0	—	421·5	—	348·0	—	325·0	—
,, ,,	,,	,,	4·2913	389·0	—	384·5	—	311·0		308·5	—	285·0
,, ,,	,,	,,	4·2931	361·5	—	358·0	—	295·0	—	286·0	—	268·0
,, ,,	,,	,,	4·2940	420·0	—	415·0	—	347·0	—	333·0	—	321·0
,, ,,	,,	,,	4·2943	420·0	—	420·0	—	—	337·0	—	—	—
,, ,,	,,	,,	4·2951	422·0	—	414·5	—	344·0	—	325·0	—	307·5
,, ,,	Londesborough	U.M.O.	E. 11·9 641	—	459·5	—	456·5	—	—	—	—	—
,, Cambs.	Burwell	C.D.A.	X 1	466·3	460·4	462·6	455·0	367·3	—	—	—	318·1
,, ,,	,,	,,	3	433·8	432·3	427·5	426·9	365·4	364·4	—	—	—
,, ,,	,,	,,	13	426·7	426·2	423·0	419·8	346·3	345·1	—	—	314·4
,, ,,	,,	,,	32	398·0	399·9	392·9	394·0	327·0	328·8	—	—	299·3
,, ,,	,,	,,	40	419·9	426·4	419·0	425·3	341·1	340·2	—	—	—
,, ,,	,,	,,	59	431·7	433·0	425·0	430·0	—	342·4	—	—	—
,, ,,	,,	,,	80	417·2	417·2	413·0	413·0	336·8	336·8	329·0	—	306·0
,, ,,	,,	,,	119	430·4	438·0	428·0	436·0	334·8	342·0	—	—	304·5
W.S. Oxon.	Dorchester	U.M.O.	E. 11·9 609	420·0	—	415·0	—	—	338·0	—	—	300·0
,, ,,	Yarnton	,,	,, 616	401·6	—	399·0	—	342·3	344·2	—	—	—
,, ,,	Minster Lovell	,,	,, 629	—	—	—	—	383·0	383·0	—	—	—
,, ,,	Yarnton	,,	,, 640	448·0	452·5	444·3	448·0	364·6	—	—	—	313·0
,, Berks.	Abingdon	,,	,, 14	436·0	435·0	433·0	429·0	364·0	364·5	361·5	358·0	315·0
,, ,,	,,	,,	,, 29	419·5	—	416·0	—	343·0	340·0	332·0	331·0	313·5
,, ,,	,,	,,	,, 31	—	451·0	—	447·0	380·5	383·0	—	—	334·5
,, ,,	,,	,,	,, 34	407·0	412·0	403·0	409·0	340·0	342·0	326·5	—	306·3
,, ,,	,,	,,	,, 50	423·0	421·0	419·0	418·0	351·0	350·5	342·0	—	310·5
,, ,,	,,	,,	,, 60	398·0	401·5	393·0	398·0	—	322·0	314·5	—	296·0
,, ,,	,,	,,	,, 66	—	463·5	—	460·5	—	381·0	—	—	—
,, ,,	,,	,,	,, 75	—	386·0	—	383·5	314·0	316·0	—	—	294·0
,, ,,	,,	,,	,, 78	421·0	425·5	417·0	422·5	349·0	349·0	—	—	—
,, ,,	,,	,,	,, 84	—	461·5	—	450·0	375·5	381·0	—	—	328·0
,, ,,	,,	,,	,, 85	465·0	464·0	458·5	459·5	380·0	—	—	—	335·0
,, ,,	,,	,,	,, 92	458·5	461·4	455·0	456·0	370·5	371·0	358·1	358·7	332·8
,, ,,	,,	,,	,, 103	453·5	454·0	448·8	449·4	—	367·2	—	—	338·6
,, ,,	,,	,,	,, 104	446·0	447·0	443·0	442·9	—	374·7	—	—	—
,, ,,	,,	,,	,, 106	438·1	441·0	438·1	441·0	361·0	364·1	—	351·3	315·8
,, ,,	,,	,,	,, 113	410·8	408·1	406·3	408·0	322·8	321·6	—	—	292·0
,, ,,	,,	,,	,, 118	412·2	416·2	410·3	412·1	339·4	338·0	327·5	326·1	308·3
,, ,,	,,	,,	,, 119	433·0	436·0	429·4	430·3	360·8	361·0	—	—	316·0
,, ,,	Sutton Courtenay	,,	,, 583	429·0	430·0	425·6	425·0	349·5	350·5	—	—	313·8
,, ,,	Wytham	,,	,, 613	426·7	428·8	423·0	426·8	—	—	—	—	—
,, ,,	,,	,,	,, 644	—	412·2	—	406·6	—	—	—	—	—
,, ,,	East Shefford	R.C.S.	4·2449	420·0	420·0	414·0	411·0	340·0	338·5	—	—	291·0
,, ,,	,,	,,	4·2455	—	—	—	—	369·0	—	—	—	—
,, ,,	,,	,,	4·2460	408·0	411·5	402·0	406·0	343·0	341·0	—	—	293·0
,, Wilts.	Winkelbury Hill	P.R.M.	1	404·0	396·0	—	—	351·0	341·0	—	—	—
,, ,,	,,	,,	7	422·0	—	—	—	362·0	—	—	—	313·0
,, ,,	,,	,,	23	430·0	427·0	—	—	347·0	349·0	—	—	318·0
,, ,,	,,	,,	24	—	411·0	—	—	—	—	—	—	—
,, ,,	Roche Court Down	R.C.S.	4·2522	422·0	422·5	419·0	418·0	—	—	—	—	—
,, ,,	,,	,,	4·2526	425·0	429·0	422·5	428·0	347·0	351·0	—	—	328·0
,, ,,	,,	,,	4·2527	422·0	—	419·5	—	—	349·0	—	—	—
,, ,,	,,	,,	4·2528	—	442·0	—	436·5	—	370·0	—	—	—
S.S. Surrey	Mitcham	,,	4·2645	—	—	—	—	375·0	371·0	—	—	—
,, ,,	,,	,,	4·2663	458·0	456·0	455·0	454·0	—	384·5	—	—	324·0
,, ,,	Guildown	,,	4·2710	472·0	—	468·5	466·0	380·0	—	—	—	334·0
J. Hants.	Meon Hill	,,	4·2547	453·5	459·0	449·0	454·0	363·0	360·0	357·5	352·0	326·0

E XX. *Individual Measurements of Female Anglo-Saxon Long Bones.*

	Fibulae		Humeri				Radii		Ulnae		100 × Radius Max. / Humerus Max.		100 × Humerus / Femur	
	Maximum		Maximum		Oblique		Maximum		Maximum					
	R	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	
·0	—	325·0	—	297·0	—	292·0	—	224·0	—	247·0	—	75·42	—	7
	308·5		285·0	—	281·0	—	207·0	—	227·5	—	72·63	—	74·12	
	286·0	—	268·0	—	265·0	—	195·0	—	214·0	—	72·76	—	74·86	
	333·0	—	321·0	—	314·0	—	235·0	—	247·0	—	73·21	—	77·35	
·0	—	—	—	—	—	—	—	—	—	—	—	—	—	
	325·0		307·5		302·0	—	227·0	—	243·0		73·82	—	74·19	
	—	—	—	323·5	—	314·8	—	—	—	—	—	—	—	7
	—	—	318·1	—	311·9	—	—	239·5	—	257·4	—	—	68·76	
·4	—	—	—	—	—	—	—	—	—	—	—	—	—	
·1	—	—	314·4	311·7	306·9	308·4	—	—	—	—	—	—	74·33	74
·8	—	—	299·3	—	297·4	—	—	—	—	—	—	—	76·18	
·2	—	—	—	—	—	—	—	—	—	—	—	—	—	
·4	—	—	—	310·8	—	308·4	—	224·0	—	245·7	—	72·07	—	72
·8	329·0	—	306·0	—	302·6	—	213·3	214·0	236·8	—	69·71	—	74·09	
·0	—	—	304·5	—	302·5	—	220·0	—	245·0	—	72·25	—	71·14	
·0	—	—	300·0	289·0	296·0	283·5	—	219·0	—	242·0	—	75·78	72·29	
·2	—	—	—	303·1	—	298·8	—	—	—	—	—	—	—	
·0	—	—	—	—	—	—	—	—	—	—	—	—	—	
	—	—	313·0	—	306·8	—	—	—	—	—	—	—	70·45	
·5	361·5	358·0	315·0	311·0	314·0	310·5	233·0	229·0	254·0	247·5	73·97	73·63	72·75	72
·0	332·0	331·0	313·5	307·0	312·5	300·5	239·5	—	266·0	—	76·40	—	75·36	
·0	—	—	334·5	325·0	333·5	324·0	—	238·0	262·5	—	—	73·23	—	72
·0	326·5	—	306·0	302·0	302·5	300·0	221·0	221·0	241·0	239·0	72·22	73·18	75·93	73
·3	342·0	—	310·5	301·0	309·0	299·0	234·0	231·5	—	255·0	75·36	76·91	74·11	72
·0	314·5	—	296·0	291·5	291·5	287·5	211·0	—	232·5	—	71·28	—	75·32	73
·0	—	—	—	320·5	—	317·0	—	—	—	—	—	—	—	69
·0	—	—	294·0	286·5	291·0	283·5	222·0	216·0	237·0	233·0	75·51	75·39	—	74
·0	—	—	—	321·5	—	318·0	—	—	—	—	—	—	—	76
·0	—	—	328·0	—	322·0	—	—	—	—	—	—	—	—	
	—	—	335·0	—	330·0	—	245·0	242·5	254·0	259·5	73·13	—	73·06	
·0	358·1	358·7	332·8	322·4	322·8	313·2	237·6	235·4	—	245·0	71·39	73·01	73·14	70
·2	—	—	338·6	329·0	332·0	322·6	—	—	—	—	—	—	75·45	73
·7	—	—	—	320·1	—	314·0	—	—	—	—	—	—	—	72
·1	—	351·3	315·8	312·1	315·0	310·4	237·9	237·0	—	259·0	75·33	75·94	72·08	70
·6	—	—	292·0	285·3	290·0	283·0	221·0	218·0	242·3	242·7	75·68	76·41	71·87	69
·0	327·5	326·1	308·3	305·1	299·0	296·3	236·0	235·0	256·3	252·5	76·55	77·02	75·14	74
·0	—	—	316·0	315·2	307·8	306·0	—	238·5	259·8	—	—	75·67	73·59	73
·5	—	—	313·8	312·3	307·8	306·7	—	—	—	—	—	—	73·73	73
	—	—	—	314·3	—	308·3	—	—	—	—	—	—	—	73
	—	—	—	282·5	—	281·5	—	—	—	—	—	—	—	69
·5	—	—	291·0	—	287·0	—	—	—	—	—	—	—	70·29	
	—	—	—	—	—	—	—	—	—	—	—	—	—	
·0	—	—	299·0	—	295·0	—	218·0	218·5	—	240·0	72·91	—	74·38	
·0	—	—	—	—	—	—	—	—	—	—	—	—	—	
	—	—	313·0	—	—	—	—	230·0	249·0	—	—	—	—	
·0	—	—	318·0	310·0	—	—	227·0	233·0	257·0	—	71·38	75·16	—	
	—	—	—	314·0	—	—	—	—	—	—	—	—	—	
	—	—	—	307·5	—	300·0	—	—	—	—	—	—	—	73
·0	—	—	328·0	—	323·0	—	—	229·0	—	247·0	—	—	77·63	
·0	—	—	—	—	—	—	—	—	—	—	—	—	—	
·0	—	—	—	316·5	—	313·5	—	—	—	—	—	—	—	72
·0	—	—	—	—	—	—	—	—	—	—	—	—	—	
·5	—	—	324·0	—	319·0	—	—	—	—	—	—	—	71·21	
·0	—	—	334·0	331·0	331·0	328·0	249·0	246·0	268·0	270·0	74·55	74·32	71·29	71
	357·5	352·0	326·0	328·0	324·0	325·0	233·0	234·5	254·5	254·5	71·47	71·49	72·61	72

	$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{Tibia Obl.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{(Humerus Max.} + \text{Radius Max.)}}{\text{(Femur Obl.} + \text{Tibia Max.)}}$		Angle of Femur	
	R	*L*	*R*	*L*	*R*	*L*	*R*	*L*
	—	70·46	—	81·38	—	68·02	—	7°·4
	74·12	—	79·06	—	70·94	—	8°·7	—
	74·86	—	80·45	—	71·12	—	8°·0	—
	77·35	—	81·69	—	73·25	—	8°·8	—
	—	—	—	—	—	—	0°·0	—
	74·19	—	81·30	—	70·65	—	10°·8	—
	—	70·87	—	—	—	—	—	6°·6
	68·76	—	77·89	—	—	—	7°·2	8°·8
	—	—	83·84	83·72	—	—	9°·8	9°·1
	74·33	74·25	80·21	80·54	—	—	7°·5	9°·9
	76·18	—	81·45	81·68	—	—	9°·2	9°·9
	—	—	79·74	78·34	—	—	3°·7	4°·1
	—	72·28	—	78·00	—	69·51	10°·1	6°·8
	74·09	—	79·85	79·85	69·54	—	8°·2	8°·2
	71·14	—	76·59	76·83	69·03	—	6°·1	5°·5
	72·29	—	—	—	—	—	8°·8	—
	—	—	84·04	—	—	—	6°·5	—
	—	—	—	—	—	—	—	—
	70·45	—	80·49	—	—	—	7°·3	8°·1
	72·75	72·49	82·45	83·33	68·45	68·82	6°·7	9°·5
	75·36	—	80·77	—	73·15	—	7°·4	—
	—	72·71	—	84·12	—	68·08	—	7°·6
	75·93	73·84	82·63	81·91	71·22	69·92	8°·0	6°·9
	74·11	72·01	82·10	82·18	70·93	69·56	7°·9	6°·8
	75·32	73·24	—	79·15	—	—	9°·1	7°·6
	—	69·60	—	81·22	—	—	—	6°·5
	—	74·71	—	80·57	—	72·15	—	6°·5
	—	76·09	82·01	81·04	—	—	7°·9	7°·4
	—	—	—	83·11	—	—	—	12°·8
	73·06	—	81·35	—	69·42	—	9°·6	8°·0
	73·14	70·70	80·11	79·82	69·35	67·69	7°·1	8°·8
	75·45	73·21	—	80·15	—	—	8°·3	8°·1
	—	72·27	—	83·02	—	—	6°·7	7°·8
	72·08	70·77	80·80	80·98	69·55	68·46	0°·0	0°·0
	71·87	69·93	77·72	77·11	70·65	69·27	8°·5	1°·1
	75·14	74·04	81·01	80·32	72·89	72·29	5°·5	8°·0
	73·59	73·27	82·39	82·27	—	70·24	7°·4	9°·3
	73·73	73·48	80·47	80·82	—	—	7°·2	8°·8
	—	73·64	—	—	—	—	7°·5	5°·5
	—	69·48	—	—	—	—	—	9°·5
	70·29	—	80·43	80·78	—	—	9°·7	11°·9
	—	—	—	—	—	—	—	—
	74·38	—	83·58	82·02	69·49	—	9°·8	9°·4
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—
	—	73·56	—	—	—	—	6°·8	8°·4
	77·63	—	80·83	80·37	—	—	6°·2	3°·9
	—	—	—	—	—	—	6°·3	—
	—	72·51	—	83·51	—	—	—	9°·1
	—	—	—	—	—	—	—	—
	71·21	—	—	83·04	—	—	6°·6	5°·4
	71·29	71·03	79·40	—	68·95	—	7°·0	—
	72·61	72·25	79·06	77·64	69·10	69·36	8°·1	8°·5

UNIV. OF MICH.

TABLE XXI. *I*

Division and County	Cemetery	Museum	Museum No.	Femora				Tibiae		Fi
				Maximum		Oblique		Maximum		Max
				R	*L*	*R*	*L*	*R*	*L*	*R*
A. Yorks.	Scarborough	R.C.S.	4·2902	—	—	—	—	—	326·0	—
,, ,,	,,	,,	4·2908	424·0	—	422·0	—	361·0	—	347·0
,, ,,	,,	,,	4·2916	409·0	—	406·5	—	332·0	—	321·0
,, ,,	,,	,,	4·2926	—	—	—	—	—	—	—
,, ,,	,,	,,	4·2927	—	424·0	—	417·0	—	329·0	—
,, ,,	,,	,,	4·2930	—	419·0	—	415·0	—	—	—
,, ,,	,,	,,	4·2934	—	413·0	—	405·0	—	356·0	—
,, Cambs.	Burwell	C.D.A.	2	430·8	436·0	428·1	432·0	363·1	366·3	—
,, ,,	,,	,,	25	417·2	412·1	412·0	410·4	338·2	338·9	—
,, ,,	,,	,,	36	430·1	—	422·7	—	—	355·5	—
,, ,,	,,	,,	98	417·6	416·0	415·6	414·2	334·3	338·2	—
,, ,,	War Ditches	,,	—	417·3	419·4	412·0	413·9	—	336·0	—
,, Norfolk	Thetford	U.M.O.	E. 11·9 623	—	426·0	—	423·0	—	—	—
W.S. Oxon.	Cassington	,,	,, 585a	454·6	449·0	448·4	443·5	360·5	360·6	344·2
,, Berks.	Abingdon	,,	,, 6	456·0	441·5	451·0	441·0	—	376·5	—
,, ,,	,,	,,	,, 32	—	—	—	—	—	—	—
,, ,,	,,	,,	,, 53	404·0	—	397·0	398·0	—	—	—
,, ,,	,,	,,	,, 101	467·0	464·2	461·5	458·6	—	—	—
,, ,,	,,	,,	,, 102	—	—	—	—	—	—	—
,, ,,	,,	,,	,, 105	—	—	—	—	—	—	—
,, ,,	,,	,,	,, 109	462·8	470·0	460·6	465·0	369·3	369·4	—
,, ,,	,,	,,	,, 114	—	390·5	—	387·0	—	—	—
,, ,,	,,	,,	,, 117	—	—	—	—	—	—	—
,, ,,	Wytham	,,	,, 627	—	—	—	—	—	—	—
,, ,,	East Shefford	R.C.S.	4·2448	—	—	—	—	—	—	—
,, ,,	,,	,,	4·2451	395·0	394·0	392·5	391·0	336·5	—	—
,, Wilts.	Winkelbury Hill	P.R.M.	20	—	453·0	—	—	—	368·0	—
S.S. Surrey	Mitcham	R.C.S.	4·2655	—	—	—	—	—	—	—

E XXI. *Individual Measurements of Female Anglo-Saxon Long Bones* (c

ibiae		Fibulae		Humeri				Radii		Ulnae		100 × Radius / Humerus
ximum		Maximum		Maximum		Oblique		Maximum		Maximum		
	L	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*
	326·0	—	—	—	292·0	—	283·5	—	212·5	—	—	—
	—	347·0	—	328·0	320·0	321·0	314·0	—	—	—	—	—
	—	321·0	—	—	—	—	—	—	215·0	—	234·5	—
	—	—	—	325·0	—	320·0	—	236·0	—	253·5	—	72·62
	329·0	—	319·0	—	—	—	302·0	—	—	—	241·0	—
	—	—	—	—	301·0	—	294·0	—	216·0	—	233·0	—
	356·0	—	—	—	307·0	—	293·0	—	225·0	—	—	—
	366·3	—	—	308·0	—	304·7	—	—	—	—	252·1	—
	338·9	—	—	309·1	304·0	304·3	301·2	227	223·2	—	—	73·70
	355·5	—	—	—	—	—	—	237·0	—	—	—	—
	338·2	—	—	304·0	302·1	297·3	297·0	222·2	—	—	—	73·09
	336·0	—	—	—	309·4	—	302·0	—	228·0	—	—	—
	—	—	—	306·2	—	302·1	—	212·0	—	232·0	—	69·24
	360·6	344·2	—	338·0	332·6	333·2	326·0	241·8	—	—	—	71·54
	376·5	—	—	—	335·0	—	324·5	—	241·5	—	—	—
	—	—	—	312·0	310·5	304·0	303·0	233·0	—	—	—	74·68
	—	—	—	—	290·5	—	288·0	—	207·5	—	—	—
	—	—	—	318·5	308·0	313·4	300·0	—	245·0	268·8	264·0	—
	—	—	—	—	308·0	—	306·3	—	226·1	—	—	—
	—	—	—	335·5	329·7	328·1	325·0	253·2	249·0	—	268·0	75·47
	369·4	—	—	—	—	—	—	222·0	—	237·4	—	—
	—	—	—	288·8	284·6	286·8	280·8	212·0	206·9	233·0	—	73·41
	—	—	—	330·5	—	322·2	—	240·4	235·0	261·0	—	72·74
	—	—	—	—	—	—	—	218·1	217·4	—	—	—
	—	—	—	291·0	—	289·0	—	219·0	—	—	233·0	75·26
	—	—	318·5	—	—	—	—	—	—	—	—	—
	368·0	—	—	341·0	—	—	—	244·0	245·0	—	—	71·55
	—	—	—	303·0	299·0	300·0	298·0	230·0	—	252·0	—	75·91

Anglo-Saxon Long Bones (cont.).

dii	Ulnae		$100 \times \frac{\text{Radius Max.}}{\text{Humerus Max.}}$		$100 \times \frac{\text{Humerus Max.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{Tibia Obl.}}{\text{Femur Obl.}}$		$100 \times \frac{\text{(Humerus Max.} + \text{Radius Max.)}}{\text{(Femur Obl.} + \text{Tibia Max.)}}$		Angle of Femur	
mum	Maximum											
L	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*	*R*	*L*
212·5	—	—	—	72·77	—	—	—	—	—	—	—	—
—	—	—	—	—	77·73	—	83·65	—	—	—	5°·5	—
215·0	—	234·5	—	—	—	—	79·83	—	—	—	6°·4	—
—	253·5	—	72·62	—	—	—	—	—	—	—	—	—
—	—	241·0	—	—	—	—	—	77·22	—	—	—	10°·4
216·0	—	233·0	—	71·76	—	72·53	—	—	—	—	—	7°·9
225·0	—	—	—	73·29	—	75·80	—	86·67	—	70·18	—	11°·3
—	—	252·1	—	—	71·95	—	83·18	83·17	—	—	6°·6	7°·8
223·2	—	—	73·70	73·42	75·02	74·07	80·39	80·87	71·85	70·64	9°·1	5°·2
—	—	—	—	—	—	—	—	—	—	—	10°·6	—
—	—	—	73·09	—	73·15	72·94	78·75	79·96	70·45	—	6°·2	5°·3
228·0	—	—	—	73·69	—	74·75	—	79·49	—	71·95	9°·2	9°·3
—	232·0	—	69·24	—	—	—	—	—	—	—	—	6°·8
—	—	—	71·54	—	75·38	74·99	78·84	79·73	71·94	—	9°·5	9°·0
241·5	—	—	—	72·09	—	75·96	—	83·79	—	70·78	8°·5	2°·7
—	—	—	74·68	—	—	—	—	—	—	—	—	—
207·5	—	—	—	71·43	—	72·99	—	—	—	—	10°·7	—
245·0	268·8	264·0	—	79·55	69·01	67·16	—	—	—	—	8°·8	8°·9
226·1	—	—	—	73·41	—	—	—	—	—	—	—	—
249·0	—	268·0	75·47	75·52	—	—	—	—	—	—	—	—
—	237·4	—	—	—	—	—	78·66	77·94	—	—	5°·6	8°·4
206·9	233·0	—	73·41	72·70	—	73·54	—	—	—	—	—	7°·6
235·0	261·0	—	72·74	—	—	—	—	—	—	—	—	—
217·4	—	—	—	—	—	—	—	—	—	—	—	—
—	—	233·0	75·26	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	83·57	—	—	—	6°·5	7°·1
245·0	—	—	71·55	—	—	—	—	—	—	—	—	—
—	252·0	—	75·91	—	—	—	—	—	—	—	—	—

humerus there is clear indication that females are more asymmetrical than males. The same is probably true for the radius, with a lesser differentiation of the sexes than in the case of the humerus. The individual bilateral differences for the humerus and femur are found to be uncorrelated.

(*c*) The sex ratios (male mean/female mean) are closely similar for the different races and the different bones (Table IX), but they show a clear tendency to be slightly greater for the radius than for the other three bones. For the living series the sex ratios are greater for arm lengths and span than for leg lengths and stature.

(*d*) The coefficients of variation lead to a far closer approach to equality than is found for the standard deviations in the case of comparisons between different bones, between the two sexes, and between different series. Hence it is most convenient and suggestive to consider only the measures of variability relative to size (Table X). The male coefficients show a slight tendency to be greater than the corresponding female values for the material considered, but there is less indication of a differentiation of the two sides in the respect considered. In the case of a particular series the coefficients of variation for the four bones are closely similar to one another, but there is a slight suggestion that the tibia and radius (i.e. the lower bones of the members) tend to be relatively more variable than the femur and humerus. Comparisons between six different series also reveal a remarkably close approach to equality in variation, and there is no suggestion that those representing "primitive" races are either more or less variable than those representing civilised races.

(*e*) Correlations between the lengths of different bones are given in Tables XI—XIII. Few female values were calculated, as there is little suitable material for this sex, and these few all differ insignificantly from the corresponding male values. It is probable that the conclusions below—derived from male constants—could be derived also from series of female skeletons. The correlations can be divided into three classes:

(α) the bilateral, of the order ·98 (Table XI);

(β) those, other than the bilateral, between different bones of the same member, whether on the same or opposite sides, these being, with very few exceptions, less than the bilateral and of the order ·87 (Table XII);

(γ) those between bones of different members, whether on the same or opposite sides, these being, in general, with some exceptions, less than those of class (β) and of the order ·83 (Table XIII).

These generalisations are, at any rate, close approximations to the true state of affairs in the case of the series considered, and it is surprising to find that they can be made, since they are of such a nature that for each class considered separately right and left may be interchanged, and the different bones may also be interchanged, without affecting the order of the correlations to more than a slight extent, if at all significantly. A few unimportant qualifications must be made. There

is a slight suggestion that the radius tends to give lower bilateral correlations than the other three bones, and that the femur and tibia, being undifferentiated from one another in the respect considered, tend to give higher values than the humerus. It is not necessary to qualify the statement made with regard to the correlations of the class (β). For such, there is no suggestion of a distinction between those for bones A and B belonging to the same side and those for bone A on one side and bone B on the other, or between correlations for two leg bones and those for two arm bones. In the class (γ) it appears that the sides can be interchanged in all possible ways without affecting the degree of association found, but the humerus-femur correlations tend to be the greatest of the four combinations, and the radius-femur correlations tend to be the smallest. While these relationships are true for every series considered, there is evidence of racial differences between the correlations. The orders given above for the three classes are actually the means for the male Anglo-Saxon series. For a French the means are ·98, ·82 and ·80, respectively, so the two European series are in close agreement. For a Predynastic Egyptian, however, the means are ·95 (some individual correlations being significantly *less* than the corresponding values for both European series), ·93 (some being significantly *greater* than those for one or other of the European series) and ·87 (no significant differences found), respectively. A few of the correlations can be given for a Chinese series and these tend to be lower than the others. These systems of correlations for different races thus appear to show appreciable differences.

(f) Only two indices expressing the proportions of the lengths of different bones are considered (Tables XIV and XV)—viz. the radius-humerus and humerus-femur—as there is little comparative material for the others sometimes given. For the first there is no clear indication of a distinction between the sides, but the male mean indices are greater than the female for all races. Some significant differences are found between the means for different European series but they cover a small range, while the means for nearly all the non-European are decidedly greater. The material available suggests that the radius-humerus index is one of those characters—such as skin colour and the degrees of projection of the jaws and nasal bridge—which tend to be constant for races belonging to the same family of races, and which make absolute distinctions between some of the different families. In the case of the humerus-femur index there is no clear evidence of sex differentiation, but means for the right side exceed those for the left. Insignificant differences are found between the means for different European races, but there is little distinction between the range for these and that for the few non-European races represented.

(g) Certain inferences regarding growth of the limb bones may be deduced from the relations summarised above. The most potent "factor" is clearly that which induces the very high bilateral correlations. It is obvious, too, that there is a general growth "factor" responsible for the correlations, which are still of a high order, between all possible pairs of bones. Since these are slightly, but consistently, higher for pairs of arm bones and pairs of leg bones than for an arm bone with a

leg bone, it is necessary to suppose that there are also "factors" peculiar to the arms and legs. No attempt is made here to estimate the nature of these "factors," but it may be noted that, from all the different kinds of statistical relations considered together, there is no suggestion that any one bone (such as the radius) is distinguished from all the others, as it would be expected to be if it were affected by use while the others were not. To be effective, a more complete statistical analysis of the problem would require a larger sample of skeletons giving lengths of all the bones than any available yet.

APPENDIX.

TABLES (XVI—XXI) OF INDIVIDUAL MEASUREMENTS OF ANGLO-SAXON LONG BONES.

The divisions denoted by letters in the first columns of the tables refer to Angles (A.), East Saxons (E.S.), West Saxons (W.S.), South Saxons (S.S.) and Jutes (J.). References to the archaeological reports on the cemeteries will be given in a later paper on the skull measurements. The museums in which the skeletons are housed are those of the Royal College of Surgeons (R.C.S.), of the Department of Human Anatomy, the University Museum, Oxford (U.M.O), of the Department of Anatomy, Cambridge (C.D.A.) and the Pitt-Rivers Museum at Farnham (P.R.M.). All the skeletons were measured by the writer of the paper except those at Farnham, and Dr Dudley Buxton's readings for these are given in the tables, but he did not record oblique lengths of the femur and humerus. Dashes for all the skeletons other than those at Farnham signify that the bone in question was missing or too imperfect to give the length.

The measurements of the skeletons at Farnham were not available until after a large part of the calculations for the Anglo-Saxon material had been completed, and they were not used in calculating the majority of the constants given in tables in the text. The following exceptions should be noted: Table II, lengths of the Farnham male radii were used in finding the bilateral correlations, but the Farnham bones were omitted in finding all the other correlations in the table; Table III, the correlations between the lengths of the radius and the maximum lengths of the humerus include the Farnham skeletons, but these were excluded in calculating all the other correlations in the table; Table V, the measurements of the Farnham bones were included in calculating all the correlations in the table; Table VII, the Farnham bones were included in calculating all the means used in the reconstruction formulae. The lengths are recorded in tenths of mm., but it is clear that owing to inconsistency in measuring there are more 0's and 5's among the final figures than there ought to be. This blemish, however, is of no practical importance since all the standard deviations of the measurements are greater than 10 mm.

Owing to the high correlations between the lengths of the different bones, it was felt that comparisons of regional groups could be restricted to one single

length. Constants found for the maximum length of the right femur are given below for four groups of the male bones (including those at Farnham), and as these show insignificant differences the pooling of the material seems to be justified.

	Means	Standard Deviations
Angles ...	460·9 ± 1·81 (72)	22·8 ± 1·28
West Saxons	466·5 ± 1·76 (71)	22·0 ± 1·25
South Saxons	462·6 (9)	—
Jutes ...	460·9 (9)	—

Editorial Note (see p. 265 above). The sampling distribution of the correlation coefficient, r, when the population correlation, ρ, is as high as ·95 is of course very far from normal for samples containing 50 to 100 individuals. Even when $\rho = ·90$ we have the following results (see *Tables for Statisticians and Biometricians*, Part II, p. 183):

$$n = 50,\ \beta_1(r) = ·666,\ \beta_2(r) = 4·221;\quad n = 100,\ \beta_1(r) = ·311,\ \beta_2(r) = 3·579.$$

While I do not think that Dr Münter's general conclusions, reached from a comparison of high correlations, are likely to be at fault, some caution is certainly necessary in interpreting the significance of differences in terms of their probable errors. I am not sure that the accuracy of Professor R. A. Fisher's very useful z-transformation has ever been examined for cases of $\rho \geqslant ·95$, but some investigation into this matter is being undertaken. E. S. P.

MOMENTS OF THE RATIO OF THE MEAN DEVIATION TO THE STANDARD DEVIATION FOR NORMAL SAMPLES.

By R. C. GEARY, M.Sc.

In a recent paper* it was suggested that, when the experimental technique has been such that the population mean might be presumed zero, the ratio

$$w_n = \frac{1}{\sqrt{n}} \frac{\Sigma |y_i|}{\sqrt{\Sigma y_i^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(where the sample measures are $y_1, y_2, \ldots, y_n$), which tends towards $\sqrt{2/\pi}$ in the normal case for $n = \infty$, might be used as a test of normality. The frequency distributions of w_n, for all sizes of sample, were determined. A method was indicated for reducing the general case of population mean unknown to this particular case using a certain transformation of variables. It was recognised, however, that this transformation, which was due to Helmert, had the disadvantage that it could be effected in $n'!$ ways ($n' = n+1$ being the number in the sample in the general case), each of which would, in general, yield a different value of w_n, *for the same sample.* In consequence, a new source of variability was introduced, superposed on the ordinary sampling variability; and, as E. S. Pearson† pointed out in his useful practical investigation of the efficiency of the proposed test, the labour involved in the calculation of even a single value of w_n is very considerable. Accordingly, I suggested the ratio

$$w_n'(=a)_{\ddagger} = \frac{1}{\sqrt{n'}} \sum_{i=1}^{n'} |x_i - \bar{x}| \Big/ \sqrt{\sum_{i=1}^{n'} (x_i - \bar{x})^2} \quad \ldots\ldots\ldots\ldots\ldots (2),$$

which has a unique value for each sample, and the expected value of which, for normal samples, is equal to the expected value of w_n. It was recognised that the derivation of the exact, or even the closely approximate, frequency distribution of w_n' for all sizes of sample would probably be very difficult, because of the non-independence of $x_1 - \bar{x}$, $x_2 - \bar{x}$, etc., but there were common-sense grounds for believing that for moderate-sized samples the distribution of w_n' must be very similar to that of w_n. In this connection it should be emphasised that it is only in extreme cases that non-normality in the population can be identified by any test for *small* samples; from the practical view-point it will only be necessary to

* *Biometrika*, Vol. XXVII. (1935), p. 310. † *Ibid.*, p. 337.

‡ At Professor E. S. Pearson's suggestion, it is proposed in future to use the symbol a to indicate the *population* value of the ratio and so thereby suggest its purposive affinity with the familiar criteria β_1 and β_2; further to use a, b_1 and b_2, respectively, for the *sample* values. The symbol w_n' is, however, retained in this paper for convenience of algebraic comparison of the moments and semi-invariants of w_n and w_n'. It is hoped that no confusion will be caused by this change of notation.

establish the similarity of the distributions for medium-sized samples. To do this, it is proposed to compare the values of the first four moments, and therefrom the values of $\sqrt{B_1}$ and B_2 of w_n and w_n' for normal samples of different sizes. Formulae for the moments of w_n have already been given*. To calculate the moments of w_n', set

$$d = \frac{1}{n'} \sum_{i=1}^{n'} |x_i - \bar{x}| \quad \text{.....(3),}$$

$$s^2 = \frac{1}{n} \sum_{i=1}^{n'} (x_i - \bar{x})^2 \quad \text{.....(4),}$$

whence
$$w_n' = \sqrt{\frac{n'}{n}} \frac{d}{s} \quad \text{.....(5)}$$

with
$$n' = n + 1.$$

For the present purpose the standard deviation of each of the normal sample measures $x_1, x_2, \ldots, x_{n'}$ can be taken as unity, without loss of generality.

By an argument exactly similar to that used in an earlier paper†, the statistical independence of w_n' and s, for normal random samples, is established. It follows that the pth moment of w_n', about a fixed origin, is

$$[w_n'^p] = \left(\frac{n'}{n}\right)^{\frac{p}{2}} \frac{[d^p]}{[s^p]} \quad \text{.....(6)}$$

(square brackets, [ ], indicating expected values), so that, since $[s^p]$ is known‡, it is necessary to determine $[d^p]$ only, for $p = 2$, 3 and 4.

Now the first moment of w_n', which, by choice of constant, has been made equal to the first moment of w_n, is given by

$$[w_n'] = \sqrt{\frac{n+1}{n}} \frac{[d]}{[s]} = \sqrt{\frac{2}{\pi}} e^{\frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{1}{112n^7} + \cdots} \quad \text{.....(7),}$$

and the second moment§ by

$$[d^2] = [w_n'^2] = \frac{1}{n+1} \left(1 + \frac{2}{\pi} \sqrt{n^2 - 1} + \frac{2}{\pi} \sin^{-1} \frac{1}{n}\right) \quad \text{.....(8).}$$

To find the third and fourth moments of d, set

$$d = \frac{1}{n'} \Sigma |z_i| \text{ with } z_i = x_i - \bar{x}.$$

Then

$$[d^3] = \frac{1}{n'^3} \{n' [|z_1|^3] + 3n'(n'-1)[|z_1|^2 |z_2|] + n'(n'-1)(n'-2)[|z_1||z_2||z_3|]\}$$
$$\text{.....(9)}$$

* *Biometrika*, Vol. XXVII. (1935), p. 312.
† *Biometrika*, Vol. XXV. (1933), p. 185.
‡ *Biometrika*, Vol. XXVII. (1935), p. 313.
§ *Ibid.*, p. 354.

and

$$[d^4] = \frac{1}{n'^4}\{n'[|z_1|^4] + 4n'(n'-1)[|z_1|^3|z_2|] + 3n'(n'-1)[|z_1|^2|z_2|^2]$$
$$+ 6n'(n'-1)(n'-2)[|z_1|^2|z_2||z_3|] + n'(n'-1)(n'-2)(n'-3)[|z_1||z_2||z_3||z_4|]\}$$
.....(10).

To find each of the expected values (in brackets [ ]) on the right-hand sides we require the single and joint frequency distributions of

$$z_1,\ z_2,\ z_3,\ \text{etc.},$$

each of which is normally and similarly distributed with mean zero and standard deviation $\sqrt{\frac{n}{n+1}}$, the coefficient of correlation between each pair being $-\frac{1}{n}$. These distributions are as follows:

$$\left.\begin{aligned}
(z_1) &: \left(\frac{n+1}{2\pi n}\right)^{\frac{1}{2}} e^{-\frac{n+1}{2n}z_1^2}\,\delta z_1 \\
(z_1, z_2) &: \frac{1}{2\pi}\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}} e^{-\frac{n}{2(n-1)}(z_1^2+z_2^2)-\frac{z_1 z_2}{n-1}}\,\delta z_1 \delta z_2 \\
(z_1, z_2, z_3) &: \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{n+1}{n-2}\right)^{\frac{1}{2}} e^{-\frac{n-1}{2(n-2)}(z_1^2+z_2^2+z_3^2)-\frac{1}{n-2}(z_2z_3+z_3z_1+z_1z_2)}\,\delta z_1\delta z_2\delta z_3 \\
(z_1, z_2, z_3, z_4) &: \frac{1}{(2\pi)^2}\left(\frac{n+1}{n-3}\right)^{\frac{1}{2}} e^{-\frac{n-2}{2(n-3)}(z_1^2+\ldots)-\frac{1}{n-3}(z_1z_2+\ldots)}\,\delta z_1\delta z_2\delta z_3\delta z_4
\end{aligned}\right\}$$
.....(11).

The law of formation of the constants in these formulae is evident. All the expected values, except two, in the expressions (9) and (10) can be found without particular difficulty. Thus

$$\left.\begin{aligned}
[|z_1|^3] &= 2\sqrt{\frac{2}{\pi}}\left(\frac{n}{n-1}\right)^{\frac{3}{2}} \\
[|z_1|^2|z_2|] &= \sqrt{\frac{2}{\pi}}\,\frac{(n^2+1)}{n^{\frac{1}{2}}(n+1)^{\frac{3}{2}}} \\
[|z_1|^4] &= 3\left(\frac{n}{n+1}\right)^2 \\
[|z_1|^3|z_2|] &= \frac{2}{\pi}\left(\frac{n}{n+1}\right)^2\left\{\frac{3}{n}\sin^{-1}\frac{1}{n} + \left(2+\frac{1}{n^2}\right)\frac{(n^2-1)^{\frac{1}{2}}}{n}\right\} \\
[|z_1|^2|z_2|^2] &= \left(\frac{n}{n+1}\right)^2\left(1+\frac{2}{n^2}\right) \\
[|z_1|^2|z_2||z_3|] &= \frac{2}{\pi n(n+1)^2}\left\{n(n-2)\sin^{-1}\frac{1}{n} + (n^2-1)^{\frac{1}{2}}(n^2+2)\right\}
\end{aligned}\right\}$$
...(12).

The remaining values are $[|z_1||z_2||z_3|]$ and $[|z_1||z_2||z_3||z_4|]$. The derivation of the former has some features of algebraical interest. We have

$$[|z_1||z_2||z_3|] = \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{n+1}{n-2}\right)^{\frac{1}{2}} \int_0^\infty z_1 dz_1 \int_0^\infty z_2 dz_2 \int_0^\infty z_3 dz_3 \times e^{-\frac{1}{2}\frac{n-1}{n-2}(z_1^2+z_2^2+z_3^2)} Se^{-\frac{1}{n-2}(\pm z_2 z_3 \pm z_3 z_1 \pm z_1 z_2)} \quad \ldots\ldots(13),$$

where S indicates summation to the eight terms found by affixing + or − signs in every possible way to z_1, z_2 and z_3 in the exponent. The integration will be made to depend on the solution of an integral which may be written

$$I = \int_0^\infty \int_0^\infty \int_0^\infty x_1 x_2 x_3 e^{-\frac{1}{2}\left(x_1^2+x_2^2+x_3^2-\frac{k}{3}(x_1+x_2+x_3)^2\right)} dx_1 dx_2 dx_3 \ldots\ldots(14),$$

k being a constant.

Transform the variables x_1, x_2 and x_3:

$$x_1 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \frac{y}{\sqrt{3}},$$

$$x_2 = -\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \frac{y}{\sqrt{3}},$$

$$x_3 = -\frac{2}{\sqrt{6}} y_2 + \frac{y}{\sqrt{3}},$$

followed by the orthogonal transformation

$$y_1 = r \sin\theta,$$

$$y_2 = r \cos\theta.$$

Then
$$x_1 x_2 x_3 = -\frac{y_1^2}{2}\left(-\frac{2}{\sqrt{6}} y_2 + \frac{y}{\sqrt{3}}\right) - \frac{y_2^3}{3\sqrt{6}} + \frac{y_2^2 y}{2\sqrt{3}} + \frac{y^3}{3\sqrt{3}},$$

$$x_1^2 + x_2^2 + x_3^2 = r^2 + y^2,$$

$$\tfrac{1}{3}(x_1 + x_2 + x_3)^2 = y^2,$$

$$\delta x_1 \delta x_2 \delta x_3 = r \delta\theta \delta r \delta y.$$

As a result of these transformations, the domain of integration

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0$$

is replaced by integration in concentric circles or parts of concentric circles with radius r in the equilateral triangle given by the intersection of the plane

$$x_1 + x_2 + x_3 = \sqrt{3}\, y \text{ (positive constant)}$$

with the x-axes, and with centre the median of the equilateral triangle, followed by integration with regard to y from 0 to ∞. On integration with regard to θ, and with the further change of variable

$$\cos\gamma = \frac{y}{\sqrt{2}} r,$$

the integral I is reduced to the following:

$$I=\int_0^\infty dy\,e^{-\frac{1}{2}y^2(1-k)}\left\{\frac{2\pi}{\sqrt{3}}\left(\frac{y^3}{3}-y\right)-\frac{1}{3\sqrt{3}}\int_0^{\pi/3}d\gamma\,e^{-\frac{y^2}{4\cos^2\gamma}}\right.$$
$$\left.\times\left(\frac{y^5\sin^2\gamma}{4\cos^6\gamma}\sqrt{1-4\cos^2\gamma}+\frac{3y^3}{2\cos^2\gamma}\sqrt{4\cos^2\gamma-3}-18y\right)\right\}\ldots\ldots(15),$$

which on integration becomes

$$I=\frac{1}{1-k}-\frac{4k(k-3)}{\sqrt{3}\,(3-2k)^{\frac{1}{2}}(1-k)}\tan^{-1}\sqrt{\frac{3}{3-2k}}\quad\ldots\ldots\ldots\ldots(16).$$

Now the full integral (13) may be written in the form

$$\left(\int_0^\infty\int_0^\infty\int_0^\infty-3\int_0^\infty\int_0^\infty\int_{-\infty}^0+3\int_0^\infty\int_{-\infty}^0\int_{-\infty}^0-\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0\right)=8\int_0^\infty\int_0^\infty\int_0^\infty-6\int_0^\infty\int_0^\infty\int_{-\infty}^\infty$$
$$\ldots\ldots(17),$$

each of the integrals being applied to the element

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{n+1}{n-2}\right)^{\frac{1}{2}}z_1z_2z_3\,e^{-\frac{1}{2}\frac{n-1}{n-2}(z_1^2+z_2^2+z_3^2)-\frac{1}{n-2}(z_2z_3+z_3z_1+z_1z_2)}\,dz_1dz_2dz_3.$$

The last integral on the right-hand side of (17) is easily evaluated. Finally

$$[|z_1||z_2||z_3|]=\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left\{\left(\frac{n-2}{n-1}\right)^{\frac{1}{2}}+\frac{6(n-1)}{n^{\frac{1}{2}}(n+1)^{\frac{3}{2}}}\left(\frac{\pi}{4}-\tan^{-1}\sqrt{\frac{n-2}{n}}\right)\right\}\ldots(18).$$

I have verified that the absolute term and the coefficients of n^{-1} and n^{-2} in the expansion of this expression in terms of n^{-1} agree with the corresponding coefficients in the formal expansion of (13), i.e. as used for the development of $[|z_1||z_2||z_3||z_4|]$ in the following paragraph.

I have not been successful in obtaining an exact expression for

$$[|z_1||z_2||z_3||z_4|]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

in terms of known functions. I doubt if one exists. From the practical point of view, however, this is not a serious deprivation because an expansion in powers of $(n-2)^{-1}$ to a requisite number of terms can readily be obtained. In fact, (19) may be written in the following form:

$$\frac{1}{(2\pi)^2}\left(\frac{n+1}{n-3}\right)^{\frac{1}{2}}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty z_1z_2z_3z_4\,dz_1dz_2dz_3dz_4\,e^{-\frac{n-2}{2(n-3)}(z_1^2+\ldots+z_4^2)}$$
$$\times Se^{-\frac{1}{n-3}(\pm z_1z_2\pm\ldots\pm z_3z_4)}\ldots\ldots\ldots\ldots(20),$$

where S indicates summation to the 16 terms found by affixing + or − signs to z_1, z_2, z_3 and z_4 in every possible way in the exponent. On formally expanding S and integrating we obtain, finally,

$$[|z_1||z_2||z_3||z_4|]=\frac{4}{\pi^2}\frac{(n+1)^{\frac{1}{2}}(n-3)^{\frac{1}{2}}}{(n-2)^4}\left\{1+\frac{12}{(n-2)^2}-\frac{32}{(n-2)^3}\right.$$
$$\left.+\frac{172}{(n-2)^4}-\frac{640}{(n-2)^5}+\frac{12{,}736}{5}\frac{1}{(n-2)^6}-\frac{47{,}104}{5}\frac{1}{(n-2)^7}+\ldots\right\}\ldots\ldots(21).$$

Substituting for the single and multiple moments of $|z_1|$, $|z_2|$, etc., given by (12), (18), and (21) in the formulae for the moments of d, given by (8), (9) and (10), after much tedious algebra the following expansions in n^{-1} are obtained:

$$\left.\begin{aligned}
m_2'' = [d^2] &= a^2 + (1-a^2)\frac{1}{n} + \left(\frac{3a^2-2}{2}\right)\left(\frac{1}{n^2}-\frac{1}{n^3}\right) + \left(\frac{37a^2-24}{24}\right)\left(\frac{1}{n^4}-\frac{1}{n^5}\right) \\
&\quad + \left(\frac{373a^2-240}{240}\right)\left(\frac{1}{n^6}-\frac{1}{n^7}\right) + \dots \\
m_3'' = [d^3] &= a\left\{a^2 + 3(1-a^2)\frac{1}{n} + \left(\frac{13a^2-8}{2}\right)\frac{1}{n^2} + \left(\frac{16-25a^2}{2}\right)\frac{1}{n^3}\right. \\
&\quad + \left(\frac{151a^2-96}{8}\right)\frac{1}{n^4} + \left(\frac{128-201a^2}{8}\right)\frac{1}{n^5} \\
&\quad \left. + \left(\frac{2513a^2-1600}{80}\right)\frac{1}{n^6} + \left(\frac{1920-3017a^2}{80}\right)\frac{1}{n^7} + \dots\right\} \\
m_4'' = [d^4] &= a^4 + 6a^2(1-a^2)\frac{1}{n} + (3-16a^2+20a^4)\frac{1}{n^2} \\
&\quad + (-6+45a^2-56a^4)\frac{1}{n^3} + (15-108a^2+133a^4)\frac{1}{n^4} \\
&\quad + \left(-30+\frac{845}{4}a^2-258a^4\right)\frac{1}{n^5} + (51-352a^2+427a^4)\frac{1}{n^6} + \dots
\end{aligned}\right\} \dots(22)$$

with
$$a^2 = \frac{2}{\pi} = \cdot 6366197723\dots,$$
$$a = \sqrt{\frac{2}{\pi}} = \cdot 7978845608\dots.$$

We naturally compare the foregoing expressions for the moments of d with the corresponding moments m_2', m_3' and m_4' of

$$\frac{1}{n}\sum_{i=1}^{n}|y_i|$$

(where the y_i are independent and normally distributed with mean zero and standard deviation unity) given in the earlier paper*:

$$\left.\begin{aligned}
m_2' &= a^2 + (1-a^2)\frac{1}{n} \\
m_3' &= a\left\{a^2 + 3(1-a^2)\frac{1}{n} + (2a^2-1)\frac{1}{n^2}\right\} \\
m_4' &= a^4 + 6a^2(1-a^2)\frac{1}{n} + (3-10a^2+11a^4)\frac{1}{n^2} + 2a^2(2-3a^2)\frac{1}{n^3}
\end{aligned}\right\} \dots(23),$$

* *Op. cit.*, p. 314. In conformity with the notation of the previous paper the letter a is used again on this page to denote the constant $\sqrt{2/\pi}$, but this must not be confused with the use of a below for w_a'.

and on substituting the numerical value of a we find

$$\left.\begin{aligned} m_2'' - m_2' &= -\cdot04507034\left(\frac{1}{n^2}-\frac{1}{n^3}\right)-\cdot01854452\left(\frac{1}{n^4}-\frac{1}{n^5}\right)-\cdot01058677\left(\frac{1}{n^6}-\frac{1}{n^7}\right)\\ m_3'' - m_3' &= \sqrt{\frac{2}{\pi}}\left\{-\frac{\cdot13521103}{n^2}+\frac{\cdot04225285}{n^3}+\frac{\cdot01609820}{n^4}+\frac{\cdot00492823}{n^5}\right.\\ &\qquad\left.-\frac{\cdot0021814}{n^6}+\frac{\cdot0085231}{n^7}+\ldots\right\}\\ m_4'' - m_4' &= -\frac{\cdot17215602}{n^2}-\frac{\cdot16282608}{n^3}+\frac{\cdot14793433}{n^4}\\ &\qquad-\frac{\cdot07753470}{n^5}+\frac{\cdot03357805}{n^6}+\ldots \end{aligned}\right\} \quad\ldots\ldots(24).$$

The minuteness of the numerical coefficients in (24) is most remarkable in the circumstances, and very satisfactory because it makes possible the calculation, from a few terms of the infinite series in n^{-1}, of the moments m_2'', m_3'' and m_4'', to the very high degree of accuracy required.

From (6), (7), (8), (23) and (24), we are now in a position to calculate the moments μ_1'', μ_2'', μ_3'' and μ_4'', and the semi-invariants λ_1', λ_2', λ_3' and λ_4' of w_n'. The expansions to n^{-4} are as follows:

$$\left.\begin{aligned} \mu_1'' &= \cdot797885+\frac{\cdot199471}{n}+\frac{\cdot024934}{n^2}-\frac{\cdot031168}{n^3}-\frac{\cdot008182}{n^4}+\ldots\\ \mu_2'' &= \cdot636620+\frac{\cdot363380}{n}-\frac{\cdot045070}{n^2}+\frac{\cdot045070}{n^3}-\frac{\cdot018545}{n^4}+\ldots\\ \mu_3'' &= \cdot507949+\frac{\cdot488845}{n}-\frac{\cdot145389}{n^2}+\frac{\cdot213974}{n^3}-\frac{\cdot230351}{n^4}+\ldots\\ \mu_4'' &= \cdot405285+\frac{\cdot577441}{n}-\frac{\cdot235105}{n^2}+\frac{\cdot422158}{n^3}-\frac{\cdot696382}{n^4}+\ldots \end{aligned}\right\} \quad\ldots(25).$$

From whence we obtain the semi-invariants

$$\left.\begin{aligned} \lambda_1' &= \mu_1'' \text{ (given above)}\\ \lambda_2' &= \mu_2'' - \mu_1''^2 = \frac{\cdot045070}{n}-\frac{\cdot124648}{n^2}+\frac{\cdot084859}{n^3}+\frac{\cdot006323}{n^4}+\ldots\\ \lambda_3' &= \mu_3'' - \lambda_1'\mu_2'' - 2\lambda_2'\mu_1'' = -\frac{\cdot016857}{n^2}+\frac{\cdot109849}{n^3}-\frac{\cdot241825}{n^4}+\ldots\\ \lambda_4' &= \mu_4'' - \lambda_1'\mu_3'' - 3\lambda_2'\mu_2'' - 3\lambda_3'\mu_1'' = \frac{\cdot011051}{n^3}-\frac{\cdot145443}{n^4}+\ldots \end{aligned}\right\} \quad\ldots(26).$$

The approximate values of $\sqrt{B_1}$ and B_2 of w_n' are given by the following formulae:

$$\sqrt{B_1} = \frac{\lambda_3'}{\lambda_2'^{\frac{3}{2}}} = -\frac{1\cdot7618}{\sqrt{n}}\left(1-\frac{2\cdot3681}{n}-\frac{8\cdot8646}{n^2}+\ldots\right)\ldots\ldots\ldots\ldots(27),$$

$$B_2 - 3 = \frac{\lambda_4'}{\lambda_2'^2} = \frac{5\cdot441}{n}\left(1-\frac{7\cdot628}{n}+\ldots\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

The values of $\sqrt{B_1}$ and B_2 for $n = 5$, 10, 25, 50 and 100 as computed from approximations (27) and (28) are compared with the more accurate values, derived from (23) and (24) (in conjunction with (6), (7) and (8)), in the following table:

TABLE I.

	Value of n				
	5	10	25	50	100
Value of $\sqrt{B_1}$:					
As computed from (27)	−0·13	−0·38	−0·31	−0·24	−0·17
Actual value	−0·33	−0·41	−0·32	−0·24	−0·17
Value of B_2:					
As computed from (28)	2·43	3·13	3·15	3·09	3·05
Actual value	2·42	3·14	3·15	3·09	3·05

Except for $\sqrt{B_1}$ for $n = 5$, the correspondence is excellent. Considering that in the expansion in terms of n^{-1} only the first two terms have been used, the correspondence for B_2 is quite remarkable.

The values of the semi-invariants and the $\sqrt{B_1}$ and B_2 of w_n and w_n' for $n = 5$, 10, 25, 50 and 100 are compared in the next table:

TABLE II.

Semi-invariants of w_n and w_n'.

	Mean $= \lambda_1$	(S.D.)$^2 = \lambda_2$	λ_3	λ_4	$\sqrt{B_1}$	B_2
w_5	·835525	·00617082	−·00027074	+·00000618	−0·56	3·16
w_5'		·00470429	−·00010545	−·00001282	−0·33	2·42
w_{10}	·818049	·00375295	−·00010696	+·00000356	−0·47	3·25
w_{10}'		·00334564	−·00008024	+·00000154	−0·41	3·14
w_{25}	·805901	·00167809	−·00002250	+·00000046	−0·33	3·16
w_{25}'		·00160882	−·00002053	+·00000040	−0·32	3·15
w_{50}	·801884	·00086990	−·00000616	+·0000000720	−0·24	3·10
w_{50}'		·00085223	−·00000590	+·0000000673	−0·24	3·09
w_{100}	·799882	·00044279	−·00000161	+·0000000103	−0·17	3·05
w_{100}'		·00043832	−·00000158	+·0000000097	−0·17	3·05

In the previous paper* it has been established that even for n as low as 10 the frequency distribution of w_n is very accurately represented by the Second Approximation to the Law of Error. The marked similarity of the frequency constants for w_n and w_n', from $n = 10$ on, shown in the last two columns of the

* *Op. cit.*, p. 329.

foregoing table, leaves no room for doubt that the frequency distribution of w_n' is similarly represented, i.e. that it is approximately as follows:

$$\frac{1}{\sigma\sqrt{2\pi}}\left\{1-\frac{\sqrt{B_1}}{6}\left(\frac{3(w_n'-\lambda_1')}{\sigma}-\frac{(w_n'-\lambda_1')^3}{\sigma^3}\right)\right\}e^{-\frac{(w_n'-\lambda_1')^2}{2\sigma^2}}\delta w_n' \quad \ldots\ldots(29).$$

On this assumption, the following table has been computed:

TABLE III.

Probability Points of $a = w_n'$.

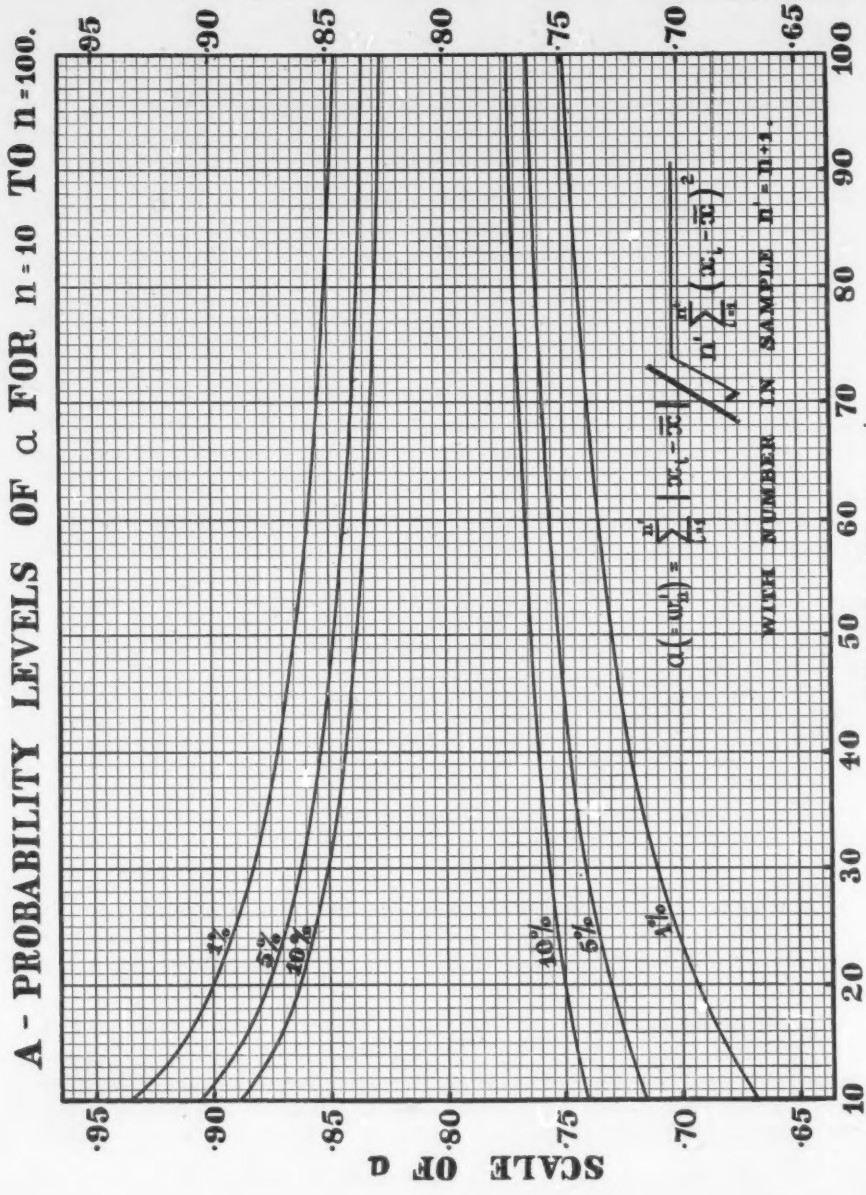
Size of Sample, n'	n	Probability Points of a						Mean of a	Standard Deviation of a
		Upper 1 %	Upper 5 %	Upper 10 %	Lower 10 %	Lower 5 %	Lower 1 %		
11	10	·9359	·9073	·8899	·7409	·7153	·6675	·81805	·05784
16	15	·9137	·8884	·8733	·7452	·7236	·6829	·81128	·04976
21	20	·9001	·8768	·8631	·7495	·7304	·6950	·80792	·04419
26	25	·8901	·8686	·8570	·7530	·7360	·7040	·80590	·04011
31	30	·8827	·8625	·8511	·7559	·7404	·7110	·80456	·03697
36	35	·8769	·8578	·8468	·7583	·7440	·7167	·80360	·03447
41	40	·8722	·8540	·8436	·7604	·7470	·7216	·80289	·03241
46	45	·8682	·8508	·8409	·7621	·7496	·7256	·80233	·03068
51	50	·8648	·8481	·8385	·7636	·7518	·7291	·80188	·02919
61	60	·8592	·8434	·8349	·7662	·7554	·7347	·80122	·02678
71	70	·8549	·8403	·8321	·7683	·7583	·7393	·80074	·02487
81	80	·8515	·8376	·8298	·7700	·7607	·7430	·80038	·02332
91	90	·8484	·8353	·8279	·7714	·7626	·7460	·80010	·02203
101	100	·8460	·8344	·8264	·7726	·7644	·7487	·79988	·02094
201	200	·8322	·8229	·8178	·7796	·7738	·7629	·79888	·01491
301	300	·8260	·8183	·8140	·7828	·7781	·7693	·79855	·01220
401	400	·8223	·8155	·8118	·7847	·7807	·7731	·79838	·01058
501	500	·8198	·8136	·8103	·7861	·7825	·7757	·79828	·00947
601	600	·8179	·8123	·8092	·7873	·7838	·7776	·79822	·00865
701	700	·8164	·8112	·8084	·7878	·7848	·7791	·79817	·00801
801	800	·8152	·8103	·8077	·7885	·7857	·7803	·79813	·00749
901	900	·8142	·8096	·8071	·7890	·7864	·7814	·79811	·00707
1001	1000	·8134	·8090	·8066	·7894	·7869	·7822	·79808	·00670

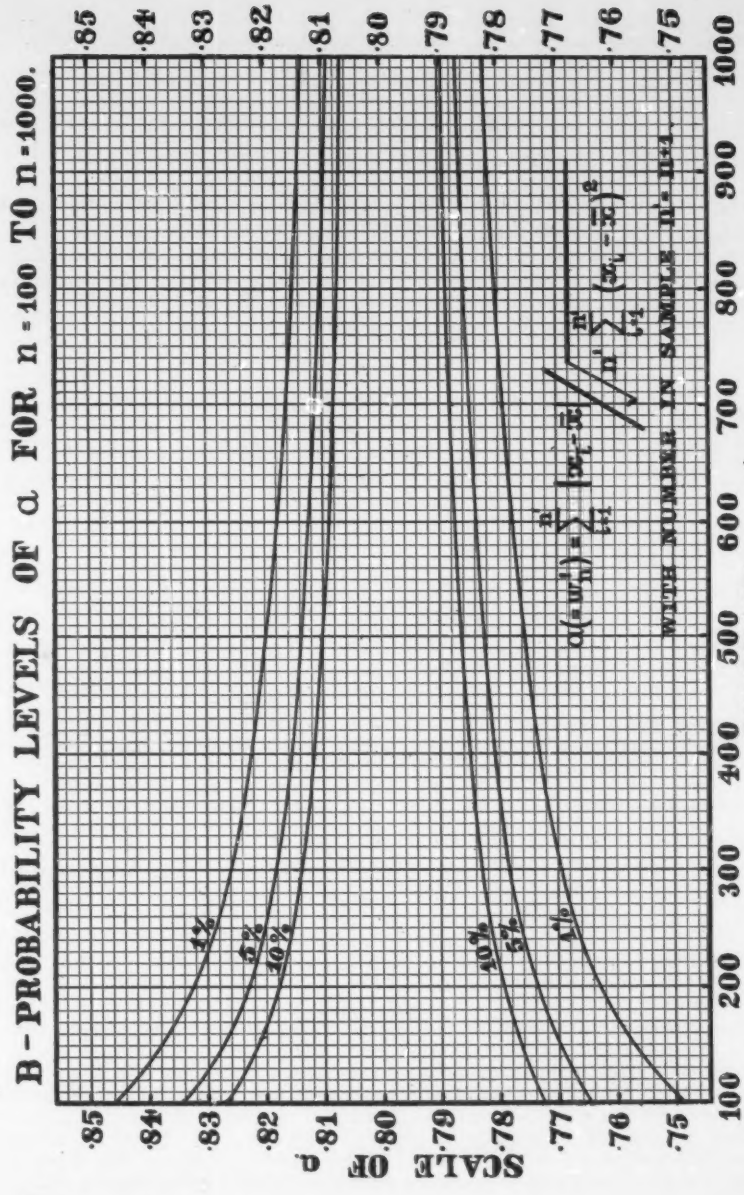
Comparing the 1 % and 5 % points with those for w_n in the earlier paper* it will be observed that, from about $n = 30$ onwards, the respective values differ by only about ·001. The "spread" is slightly less for w_n' than for w_n, for n less than 30.

In Diagrams A and B, the probability points of $a = w_n'$ for $n = 10$ to 100 and $n = 100$ to 1000 are graphed. From these diagrams it will be possible to read off the improbability of normality for all sizes of samples from 11 to 1001. The value of w_n' for large samples may be calculated from the grouped data, according to the formula given by E. S. Pearson†.

* *Op. cit.*, p. 330.

† *Op. cit.*, p. 344 (footnote).


A - PROBABILITY LEVELS OF α FOR n = 10 TO n = 100.
SCALE OF α
·95
·90
·85
·80
·75
·70
·65
1%
2½%
5%
10%
SCALE OF n = n' - 1
10
20
30
40
50
60
70
80
90
100
WITH NUMBER IN SAMPLE n' = n + 1.


B – PROBABILITY LEVELS OF α FOR n = 100 TO n = 1000.
SCALE OF ρ
SCALE OF n = n' – 1
1%
5%
10%
40%
5%
1%
WITH NUMBER IN SAMPLE n' = n + 1

NOTE ON PROBABILITY LEVELS FOR $\sqrt{b_1}$, BY E. S. PEARSON.

As pointed out by Mr Geary and myself in our earlier papers, the w_n' or a-criterion is suitable for detecting changes in kurtosis, but is not appropriate to detect asymmetry. For this purpose the most useful criterion is*

$$\sqrt{b_1}=\frac{m_3}{m_2^{\frac{3}{2}}},$$

where

$$m_2=\frac{1}{n'}\sum_{i=1}^{n'}(x_i-\bar{x})^2, \quad m_3=\frac{1}{n'}\sum_{i=1}^{n'}(x_i-\bar{x})^3.$$

To make available in one place these two tests, it seems desirable to put together here the two tables of approximate 5 °/₀ and 1 °/₀ probability points for $\sqrt{b_1}$ which have been previously issued in this Journal†. Diagrams similar to Mr Geary's Diagrams A and B are also added‡. The values of the standard deviation of $\sqrt{b_1}$ given in the table have been calculated from the formula

$$\sigma_{\sqrt{b_1}}=\sqrt{\frac{6(n'-2)}{(n'+1)(n'+3)}}.$$

Probability points for $\sqrt{b_1}$, for random samples from a Normal Population.

Size of Sample, n'	Probability Points		Standard Deviation	Size of Sample, n'	Probability Points		Standard Deviation	Size of Sample, n'	Probability Points		Standard Deviation
	5 °/₀	1 °/₀			5 °/₀	1 °/₀			5 °/₀	1 °/₀	
25	·711	1·061	·4354	200	·280	·403	·1706	1000	·127	·180	·0772
30	·661	·982	·4052	250	·251	·360	·1531	1200	·116	·165	·0705
35	·621	·921	·3804	300	·230	·329	·1400	1400	·107	·152	·0653
40	·587	·869	·3596	350	·213	·305	·1298	1600	·100	·142	·0611
45	·558	·825	·3418	400	·200	·285	·1216	1800	·095	·134	·0576
50	·533	·787	·3264	450	·188	·269	·1147	2000	·090	·127	·0547
				500	·179	·255	·1089				
60	·492	·723	·3009	550	·171	·243	·1039	2500	·080	·114	·0489
70	·459	·673	·2806	600	·163	·233	·0995	3000	·073	·104	·0447
80	·432	·631	·2638	650	·157	·224	·0956	3500	·068	·096	·0414
90	·409	·596	·2498	700	·151	·215	·0922	4000	·064	·090	·0387
100	·38[illegible]	·567	·2377	750	·146	·208	·0891	4500	·060	·085	·0365
				800	·142	·202	·0863	5000	·057	·081	·0346
125	·350	·508	·2139	850	·138	·196	·0837				
150	·321	·464	·1961	900	·134	·190	·0814				
175	·298	·430	·1820	950	·130	·185	·0792				
200	·280	·403	·1706	1000	·127	·180	·0772				

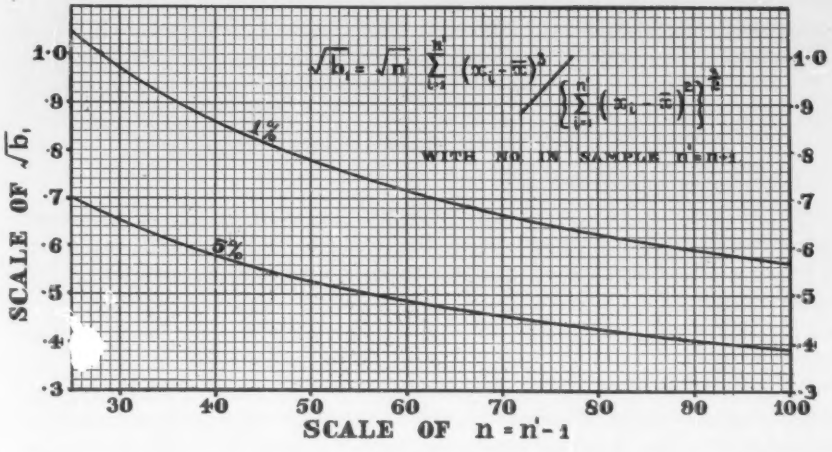
N.B.—As the sampling distribution of $\sqrt{b_1}$ is symmetrical about zero, the same values, with negative sign, correspond to the lower limits.

* The notation b_1 has been used rather than the customary β_1, to emphasise the distinction between sample and population measures.

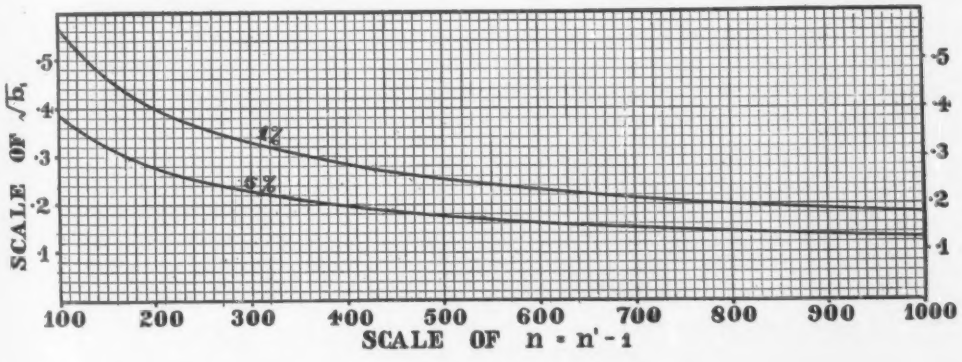
† *Biometrika*, Vol. XXII. (1931), p. 248 and Vol. XXVII. (1935), p. 271. In the latter table the third column was headed 10 °/₀ instead of 1 °/₀, through an oversight. The former table has been reproduced in *Tables for Statisticians and Biometricians*, Part II, p. 224.

‡ These four diagrams and the tables of probability points for a and $\sqrt{b_1}$, with an illustrative example, will be re-issued as a small leaflet entitled "Tests of Normality" (price *one shilling*, post free), obtainable from the *Biometrika* Office, Department of Statistics, University College, London, W.C. 1.

C - PROBABILITY LEVELS OF $\sqrt{b_1}$ FOR n = 25 - 100



D - PROBABILITY LEVELS OF $\sqrt{b_1}$ FOR n = 100 - 1000



THE EFFICIENCY OF STATISTICAL TOOLS AND A CRITERION FOR THE REJECTION OF OUTLYING OBSERVATIONS.

By E. S. PEARSON, assisted by C. CHANDRA SEKAR.

1. *Thompson's criterion.*

In an interesting paper recently published in the *Annals of Mathematical Statistics**, Dr William R. Thompson has suggested a new criterion for the rejection of outlying observations. If $x_1, x_2, \ldots x_N$ represent a series of observed values of a variable x, and

$$\bar{x} = \sum_{i=1}^{N} (x_i)/N, \quad s^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2/N \quad \ldots\ldots\ldots\ldots(1),$$

then Thompson writes

$$\tau_i = (x_i - \bar{x})/s \quad \ldots\ldots\ldots\ldots(2).$$

He then shows that if x_i is an observation arbitrarily selected from a random sample of N drawn from an infinite normal population, then the elementary probability distribution of τ is

$$p(\tau) = \frac{\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{(N-1)\pi}\,\Gamma\left(\frac{N-2}{2}\right)} \left(1 - \frac{\tau^2}{N-1}\right)^{\frac{N-4}{2}} \quad \ldots\ldots\ldots\ldots(3).$$

This is a symmetrical limited range distribution of Pearson's Type II; the probability integral of the τ-distribution may be found directly from appropriate tables†, or by noticing that if we write

$$\tau = t\sqrt{\frac{N-1}{N-2+t^2}} \quad \ldots\ldots\ldots\ldots(4),$$

then t follows Student's distribution having $N-2$ degrees of freedom, for which tables of probability levels are available‡. It will also be noted that τ is distributed as $r\sqrt{N-1}$, where r is the coefficient of correlation between N pairs of observations randomly drawn from two completely independent normal distributions.

It is clear—and consequences following from this will be discussed later—that the N values of τ given by the observations of a single sample are not independent. It is however true to say that if for any randomly chosen x_i we denote the probability that the absolute value of τ_i is greater than a specified value, say τ_0, by

$$P = P\{|\tau_i| > \tau_0\} = 2\int_{\tau_0}^{\sqrt{N-1}} p(\tau)\, d\tau \quad \ldots\ldots\ldots\ldots(5),$$

* Vol. VI. (1935), pp. 214—219.

† *Tables for Statisticians and Biometricians*, Part II, Table XXV.

‡ See Student, *Metron*, Vol. V. (1925), pp. 105—120, R. A. Fisher, *Statistical Methods for Research Workers* (1935), Table IV.

and write $\phi = NP$, then ϕ is the expectation of the number of observations per sample of N drawn from a single normal population for which

$$|x_i - \bar{x}| > \tau_0 s \quad \text{.....(6).}$$

In other words, if we decide to reject all observations for which the inequality (6) is true, then in the long run we shall reject one observation, on the average, in every $1/\phi$ samples unnecessarily, i.e. when all the observations have in fact been drawn from a common normal population.

Thompson has given a Table of the limits, τ_0, for three values of ϕ, namely 0·2, 0·1 and 0·05, and for the following values of N,

3, 4, ... 21, 22, 32, 42, 102, 202, 502, 1002.

Thus using the second value of ϕ, (0·1), and the rule of rejection denoted by the inequality (6), we shall on the average reject unnecessarily one observation per 10 samples; this figure will be the same whatever the value of N.

An alternative form of Table would be one in which limits, τ_0, were given for fixed values of $P = \phi/N$; thus ϕ would increase and $1/\phi$ decrease with the sample size. If for example $P = \cdot 01$, the observer would be likely, on the average, to discard unnecessarily one observation in every 10 samples when each of these contained 10 observations, one in every sample when it contained 100, and 10 in every sample when it contained 1000 observations. The fixing of ϕ or of P is, however, a matter of choice, the basis of Thompson's criterion being simply the equation (3) giving the probability distribution of τ.

It will be found that the great majority of criteria that have been invented for rejecting outlying observations contain as an initial condition the assumption that σ, the standard deviation in the hypothetical common population, is known*. Under certain circumstances this may be true, as for example when dealing with errors of observation where the value of σ may be estimated with great precision from past experience. Generally, however, in practice it is necessary to substitute into the formulae obtained the standard deviation calculated from the sample of observations under consideration, and this limitation, sometimes frankly recognised by the inventor of the test, sometimes apparently overlooked, renders inaccurate the probability basis upon which the criterion rests.

Dr Thompson's criterion is free from this defect; it provides, by a process which has sometimes been termed "studentizing," a true measure of the risk of rejecting an observation when in fact the whole sample has been drawn from a single normal distribution. In other words, if we describe the application of the criterion as that of testing the hypothesis, say H_0, that the sample has been drawn from a single normal population, then Thompson's method gives precise control of the risk of rejecting H_0 when it is true. This is what J. Neyman and E. S. Pearson† in their

* For an extensive survey of such tests see P. R. Rider, *Washington University Studies, Science and Technology*, No. 8 (1933).

† See, for example, *Biometrika*, Vol. XX^A. (1928), pp. 175—240, *Phil. Trans. Roy. Soc.* Vol. CCXXXI. A (1933), pp. 289—337 and *Statistical Research Memoirs* (issued by the Department of Statistics, University College, London), Vol. I. (1936), pp. 1—37.

general treatment of the theory of testing statistical hypotheses have termed the control of the first kind of error. It is, however, necessary to point out that by satisfying this condition alone it does not follow that an efficient tool has been placed in the hands of the experimenter.

This consideration has so important a bearing on the choice of statistical tests in general that it has seemed to us worth while discussing in some detail the conditions under which it appears that Thompson's method provides an efficient criterion for the rejection of outlying observations.

2. *The efficiency of statistical tools.*

What requirements should the theoretical statistician bear in mind in constructing efficient working tools for the experimenter? We think he may usefully remember two considerations which are of general application in the construction of any scientific exploratory tool:

(*a*) A tool is devised for use under certain limited conditions and will only be fully efficient as long as these conditions are satisfied.

(*b*) To test whether these conditions hold good other tools are generally needed.

An illustration of the meaning of these points from a non-statistical field may be useful. The lead-line is used at sea to measure the depth of the ocean bottom. Soundings at fixed but discrete intervals of time may be taken from a moving ship either by the hydrographer who is engaged on a survey of the ocean bottom, or by the navigator who in a fog, using the hydrographer's chart, wishes to test the hypothesis that his ship is just entering a certain channel. In both cases an assumption is made that changes in the sea bottom are gradual; if there might exist unknown pinnacled rocks of great height, sudden changes in contour or sunken ships, the lead, cast at discrete intervals of time, would be an inefficient, and even dangerous, tool to rely upon. Further, some form of dragging operation, rather than sounding, would be required to test whether it was justifiable to assume the bottom to be free from sudden changes.

The same considerations will be found to apply in the case of nearly all the tools of physical and biological science, although in many cases the conditions are so universally satisfied that the worker hardly stops to remember that limitations to the efficiency of the tool exist.

Keeping these points in mind, we may turn to the problem of the theoretical statistician who is concerned with the design of statistical tools. In the first place it must be noted that, as in other branches of applied mathematics, it is necessary for him to construct a precise but probably a simplified model which he believes will represent the phenomena of observation with sufficient accuracy to provide useful results. In so far as there is a practical problem to solve, this model will contain certain unknowns, and what is required is to devise the most effective method of obtaining information from the data concerning these unknowns. The tool will only be efficient provided that the model is appropriate, or, in other words, provided certain conditions are satisfied. To determine whether this is the case, a different set of tools will generally be required.

Some of the chief procedures of statistical analysis may usefully be classified under two heads:

(1) *The estimation of characteristics of a population*, that is to say the estimation of the values of unknowns in our model. The old procedure was to record a single-valued estimate of the unknown parameter and attach to it a probable error. More recently the conception of the confidence or fiducial interval has been introduced*. In either case the procedures employed have only a precise meaning in so far as it is possible to specify in mathematical terms the alternative forms of population distribution that are considered possible. For example, the standard error of a sample correlation coefficient, r, as ordinarily calculated, or the confidence interval, (ρ_1, ρ_2)†, have no exact meaning unless we can assume that the population distribution is of the normal form. Further, to test the validity of this assumption a different form of test is required.

(2) *The testing of statistical hypotheses.* Here the problem is to determine whether it is likely that certain unknowns in the model have specified values. In Dr Thompson's case, the problem is to test whether the sample has been drawn from a single normal population, but it is not possible to devise an efficient test if we only bring into the picture this single normal probability distribution with its two unknown parameters. We must also ask how sensitive the test is in detecting failure of the data to comply with the hypothesis tested, and to deal with this question effectively we must be able to specify the directions in which the hypothesis may fail. In other words, an efficient test of a statistical hypothesis, H_0, must be associated with a set of admissible alternative hypotheses and not solely with H_0. This set provides the model on which the statistical tool-maker can set to work.

Even when the alternatives cannot be specified in such precise form as to allow mathematical methods to be applied to full advantage, we feel sure that much is gained by a review of the types of alternatives between which it is wished to discriminate.

3. *Limitations of the τ criterion.*

Approaching from this point of view the problem of testing the hypothesis, H_0, that a sample of N observations has been drawn from a single normal population, we must ask what are the possible alternatives. Unless it is possible to conceive some alternative to H_0 there would be no justification in rejecting outlying observations. In the first place we are presumably making the assumption that the majority of observations come from some single normal population‡; we believe

* R. A. Fisher, *Proc. Camb. Phil. Soc.* Vol. XXVI. (1930), p. 528; J. Neyman, *Journ. Roy. Stat. Soc.* Vol. XCVII. (1934), p. 589; E. S. Pearson and J. C. Clopper, *Biometrika*, Vol. XXVI. (1934), pp. 404—413.

† Tables which will facilitate the calculation of these intervals prepared by Miss F. N. David will be issued shortly as a *Biometrika* Publication.

‡ If an admissible alternative to H_0 were that the observations came from a single non-normal population, then what would be required would be a test of normality, not a test for the rejection of outlying observations.

however that it is possible that one or two of them do not belong to this set. We may therefore perhaps adequately represent the situation by using the following model: alternative to H_0 are hypotheses that k of the observations ($k \geqslant 1$) come from s normal populations ($s \leqslant k$) having different means or standard deviations (or both) from the single population from which the majority ($N-k$) of the observations have been drawn. The assumption of normality in the other populations is not necessary, but as far as the present discussion goes nothing is lost by making it.

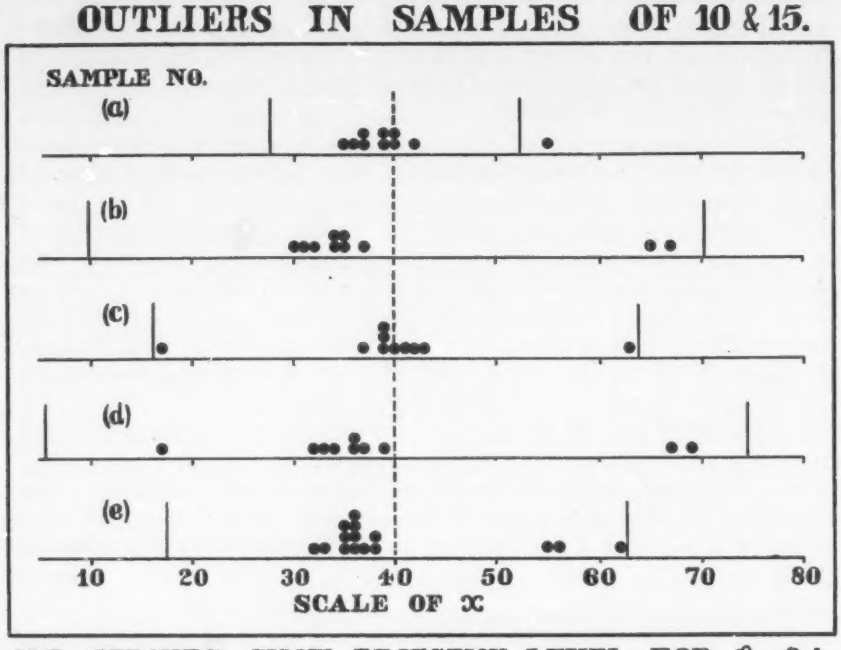


Fig. 1.

The point that we wish now to make is that if $k > 1$, Thompson's criterion is not very suitable for the purpose in view, at any rate for relatively small samples. In other words, it would appear that the criterion is only really efficient in the presence of a *single* outlying observation. To appreciate the reason for this criticism consider the data shown in Figure 1. They represent 4 different samples of 10 observations and 1 sample of 15, each having a mean of 40 on the scale shown. On intuitive grounds we should be inclined to say that in every case H_0 should be rejected, i.e. that there exist outlying observations not belonging to the main set. Yet if we apply Thompson's criterion, taking the level of significance $\phi = 0{\cdot}1$ (i.e.

running the risk of rejecting 1 observation in every 10 samples when H_0 is true), we find that it is only in the case of sample (*a*) that an outlier is picked out. In the remaining four cases none of the values of $|x_i - \bar{x}|$ exceed $\tau(\phi = 0{\cdot}1) \times s$. The limits $\bar{x} \pm \tau(\phi = 0{\cdot}1) \times s$ are marked by vertical strokes on the diagram.

We may next consider the data in Figure 2, which are based on 50 samples of 5 drawn randomly from a normal distribution with the help of Tippett's Random Sampling Numbers*.

We shall use the following notation:

(*a*) for the N values of τ in a sample arranged in descending order of absolute magnitude, write $\tau_1, \tau_2, \ldots \tau_N$; thus

$$|\tau_1| \geqslant |\tau_2| \geqslant \ldots \geqslant |\tau_N| \quad \ldots\ldots(7);$$

(*b*) for the N values arranged in magnitude taking account of sign, write $\tau^{(1)}, \tau^{(2)}, \ldots \tau^{(N)}$; thus

$$\tau^{(1)} \geqslant \tau^{(2)} \geqslant \ldots \geqslant \tau^{(N)} \quad \ldots\ldots(8).$$

Since the probability distribution for any τ taken at random follows the law of equation (3), it follows that, although the 5 values of τ obtained from the same sample are not independent, the total distribution of the $50 \times 5 = 250$ values of τ obtained in the experiment may be graduated by the curve obtained by writing $N = 5$ in the equation. This is shown in the upper portion of Figure 2. Below are shown distributions of (*A*) τ_1, (*B*) τ_2 and (*C*) τ_3, τ_4 and τ_5, and again of (*A*) $\tau^{(1)}$, (*B*) $\tau^{(2)}$, (*C*) $\tau^{(3)}$, (*D*) $\tau^{(4)}$ and (*E*) $\tau^{(5)}$.

The outer limits for τ_1 are $\pm\sqrt{N-1} = 2$; there are also outer limits for τ_2 and τ_3 which are shown by vertical strokes. Again limits exist for $\tau^{(i)}$. An explanation is at once suggested of the failure of the criterion to pick out the outlying observations in Figure 1; for certain sizes of sample the upper limit of $|\tau_i|$ $(i \geqslant 2)$ lies within the significance level given by the test. When this is so, only 1 observation can possibly be rejected however heterogeneous the data, and if the two extreme observations are close together none can be rejected at all.

To investigate this point more fully we must take the general case of a sample of N, and consider what limits there are to the value of the τ's. The problem may be put in this form:

If $\tau_1, \tau_2, \ldots \tau_N$ are any real numbers satisfying the equalities

$$\tau_1 + \tau_2 + \ldots + \tau_N = 0 \quad \ldots\ldots(9),$$

$$\tau_1^2 + \tau_2^2 + \ldots + \tau_N^2 = N \quad \ldots\ldots(10),$$

and the inequalities

$$\tau_1^2 \geqslant \tau_2^2 \geqslant \ldots \geqslant \tau_N^2 \quad \ldots\ldots(11),$$

it is required to determine the maximum value of τ_i^2 $(i = 1, 2, \ldots N)$.

* *Tracts for Computers*, XV.

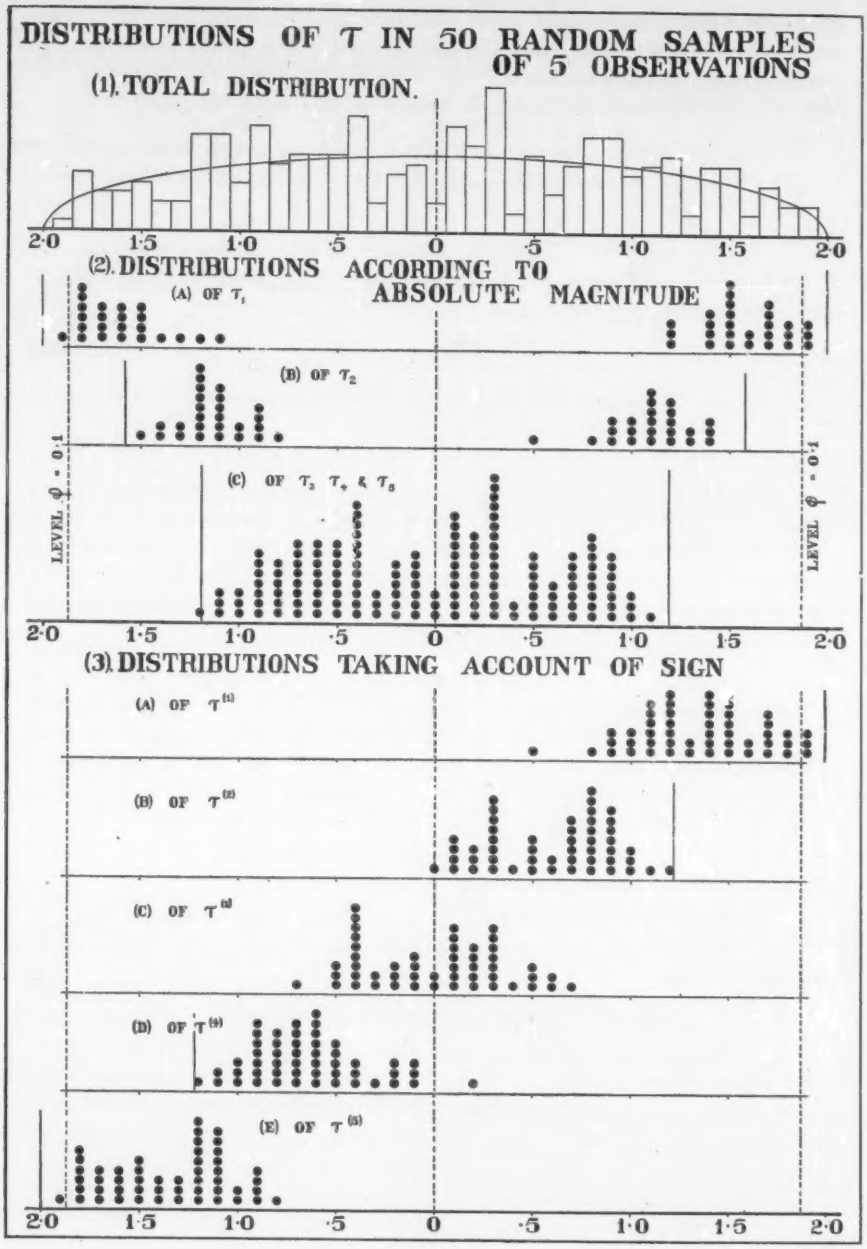

DISTRIBUTIONS OF T IN 50 RANDOM SAMPLES OF 5 OBSERVATIONS
(1). TOTAL DISTRIBUTION.
2·0 1·5 1·0 ·5 0 ·5 1·0 1·5 2·0
(2). DISTRIBUTIONS ACCORDING TO ABSOLUTE MAGNITUDE
(A) OF T_1
(B) OF T_2
(C) OF T_3 T_4 & T_5
LEVEL ϕ - 0·1
LEVEL ϕ - 0·1
2·0 1·5 1·0 ·5 0 ·5 1·0 1·5 2·0
(3). DISTRIBUTIONS TAKING ACCOUNT OF SIGN
(A) OF $T^{(1)}$
(B) OF $T^{(2)}$
(C) OF $T^{(3)}$
(D) OF $T^{(4)}$
(E) OF $T^{(5)}$
2·0 1·5 1·0 ·5 0 ·5 1·0 1·5 2·0

Fig. 2.

A detailed solution given by Mr J. M. C. Scott is set out in an Appendix to this paper. Here we shall only quote the results.

(1) If i is odd, but $< N$,

$$\text{Maximum } |\tau_i| = \sqrt{\frac{N}{i + \frac{1}{N-i}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

e.g.
$$\begin{cases} i = 1, \text{ maximum } |\tau_1| = \sqrt{N-1}, \\ i = 3, \text{ maximum } |\tau_3| = \sqrt{N(N-3)/(3N-8)}, \\ i = 5, \text{ maximum } |\tau_5| = \sqrt{N(N-5)/(5N-24)}. \end{cases}$$

(2) If $i = N$ and is odd,

$$\text{Maximum } |\tau_N| = \sqrt{\frac{N-1}{N+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

(3) If i is even, $$\text{Maximum } |\tau_i| = \sqrt{\frac{N}{i}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

e.g.
$$\begin{cases} i = 2, \text{ maximum } |\tau_2| = \sqrt{\tfrac{1}{2}N}, \\ i = 4, \text{ maximum } |\tau_4| = \sqrt{\tfrac{1}{4}N}. \end{cases}$$

(4) A further result will be useful. If we do not consider the absolute magnitude, but the maximum value of the second largest value of τ or of $\tau^{(2)}$, we find

$$\left.\begin{aligned} &\text{Maximum } \tau^{(2)} = \sqrt{\tfrac{1}{2}(N-2)} \quad \ldots\ldots\ldots\ldots \\ \text{Similarly} \quad &\text{Minimum } \tau^{(N-1)} = -\sqrt{\tfrac{1}{2}(N-2)} \quad \ldots\ldots\ldots\ldots \end{aligned}\right\} (15).$$

Putting $N = 5$, we obtain the limits shown in Figure 2.

As shown in the Appendix, the maximum value of $|\tau_1|$ occurs when $N-1$ observations have identical values, and the remaining observation any other different value; the situation in Figure 1 (a) is approaching this. The maximum $\tau^{(2)}$ occurs when $N-2$ observations have identical values and the other two have a different common value, i.e. $\tau^{(1)} = \tau^{(2)}$ (see Figure 1 (b)). The maximum $|\tau_2|$ occurs when $N-2$ observations have identical values, the other 2 differ and $\tau_1 = -\tau_2$ (see Figure 1 (c)). The maximum $|\tau_3|$ occurs when $N-3$ observations have identical values, the other 3 differ and $\tau_1 = \tau_2 = -\tau_3$ (see Figure 1 (d)).

In Figure 3 we have plotted for different values of N:

(a) the maximum values of $|\tau_i|$ ($i = 2$, 3 and 4) from equations (12) and (14);

(b) the levels of significance for τ from Thompson's tables with $\phi = 0{\cdot}20$, $0{\cdot}10$ and $0{\cdot}05$ with an additional limit $\phi = 0{\cdot}02$;

(c) the maximum value of $\tau^{(2)}$ from equation (15).

Thus if we use the significance level for ϕ of $0{\cdot}10$, involving the risk of rejection of 1 observation in every 10 samples when H_0 is true, we see that under no circumstances can we reject more than 1 observation until we reach a sample of 11; we cannot reject more than 2 observations until $N = 22$; and we cannot reject more than 3 observations until $N = 32$. This result corresponds to the extreme

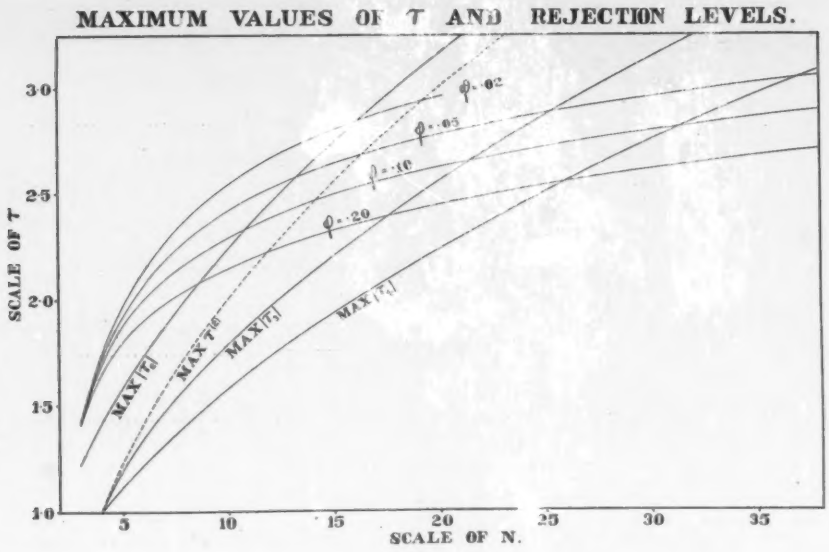


Fig. 3.

cases in which the $N-i$ observations are identical; for cases more likely to occur, such as those shown in Figure 1, the sample sizes must of course be larger.

It seems to follow, therefore, that if the situation we are faced with can be represented by a model in which the alternative to H_0 is that *one* observation has been drawn from a population with divergent mean (and possibly different standard deviation) to the remaining $N-1$ observations, Thompson's criterion will be very useful. If however the alternatives are that $k>1$ observations come from another or other systems, then the test may be quite ineffective, particularly if the sample contains not more than 30 or 40 observations.

No doubt suitable criteria could be devised for each type of alternative; thus if it were possible that 2 observations had come from a population with different mean to the remainder (Figure 1 (*b*)), the criterion might involve the calculation of the ratio of, (i) the deviation of the mean of two extreme observations from the mean of the whole to, (ii) the standard deviation. But this test would be quite unsuitable if the divergent observations had come from two populations with means diverging in opposite directions from the mean of the remainder, as in Figure 1 (*c*).

The statistician who does not know in advance with which type of alternative to H_0 he may be faced, is in the position of a carpenter who is summoned to a house to undertake a job of an unknown kind and is only able to take one tool with him! Which shall it be? Even if there is an "omnibus" tool, it is likely to be far less sensitive at any particular job than a specialised one; but the specialised tool will be quite useless under the wrong conditions.

Following the lines that Student has suggested when using "range" as a criterion*, it might be thought that a suitable "omnibus" tool for rejecting outlying observations could be obtained as follows: (1) apply Thompson's criterion to the N observations; (2) if it rejects $k \geqslant 1$ outliers, apply the criterion again to the remaining $N-k$ observations, calculating $\bar{x}$ and s afresh for the reduced sample; (3) repeat this process until a stage is reached when no further observations are rejected. Two points must however be remembered:

(*a*) Even when there are obvious outliers, the process may never get started at all for reasons already discussed. This is the case with the samples shown in Figure 1 (*b*)—(*e*).

(*b*) In order to control the first kind of error (in Neyman and Pearson's sense), i.e. the risk of rejecting H_0 when it is true, considerable development of theory would be required involving the determination of the simultaneous distribution of τ_1, τ_2, τ_3, etc.

In conclusion, since it is sometimes held that the appropriate test can be chosen *after* examining the data in the sample, a final word of caution is necessary. To base the choice of the test of a statistical hypothesis upon an inspection of the observations is a dangerous practice; a study of the configuration of a sample is almost certain to reveal some feature, or features, which are exceptional if the hypothesis is true. In the present instance it might appear, for example, that the 1st and 2nd observations (in order of magnitude) were unusually far apart, or a gap might occur between the 2nd and 3rd or between the 5th and 6th; again, the standard deviation might be large compared to the range, or there might appear to be too few observations near the mean.

By choosing the feature most unfavourable to H_0 out of a very large number of features examined, it will usually be possible to find some reason for rejecting the hypothesis. It must be remembered, however, that the point now at issue will not be whether it is exceptional to find a given criterion with so unfavourable a value. We shall need to find an answer to the more difficult question. Is it exceptional that the most unfavourable criterion of the n, say, examined should have as unfavourable a value as this?

4. *The percentage limits of the extremes, $\tau^{(1)}$ and $\tau^{(N)}$.*

An examination of Figure 2 for the case $N=5$ shows how the form of the total τ-distribution at its extreme depends only on the distributions of $\tau^{(1)}$ and $\tau^{(5)}$. This is because the upper limit of $\tau^{(2)}$ lies at $+1{\cdot}225$ and the lower limit of $\tau^{(4)}$ at $-1{\cdot}225$. In general for $\tau^{(1)} > \sqrt{\frac{1}{2}(N-2)}$ we have for the probability law of $\tau^{(1)}$,

$$p(\tau^{(1)}) = Np(\tau) \qquad \text{.....(16),}$$

where $p(\tau)$ is given in equation (3). A similar form holds for the lowest τ if $\tau^{(N)} < -\sqrt{\frac{1}{2}(N-2)}$. Since $\phi = NP$, where P is defined in equation (5), it follows that the limits of τ which Thompson has tabled for $\phi = 0{\cdot}20$, $0{\cdot}10$ and $0{\cdot}05$ will

* *Biometrika*, Vol. XIX. (1927), pp. 161—162.

correspond to the upper 10 %, 5 % and 2·5 % probability levels of $\tau^{(1)}$ (and the corresponding lower levels for $\tau^{(N)}$), as long as these levels fall beyond

$$\text{Max. } \tau^{(2)} = \sqrt{\tfrac{1}{2}(N-2)}.$$

In Figure 3 a dotted line has been drawn showing the changing value of this latter function and, by observing where it crosses the successful levels of ϕ, we can tell up to what size of sample the general distribution of τ of equation (3) may be used to give the percentage limits of the τ's calculated from the two extreme observations in a sample. Besides Thompson's levels at $\phi = 0{\cdot}20$, $0{\cdot}10$ and $0{\cdot}05$ an additional level at $\phi = 0{\cdot}02$ has been calculated*. From these we obtain the following table, in which columns 2, 3 and 4 repeat Thompson's results. While it would be useful to find the percentage limits of the extreme τ for values of N beyond those shown in the table, these cannot be obtained from the probability integral of the general τ-distribution, owing to the overlap of the distributions of $\tau^{(2)}$ and $\tau^{(1)}$ for larger values of N.

Upper probability limits for $\tau = \dfrac{x - \bar{x}}{s}$, *for the highest observation in a sample of* N, *i.e. for* $\tau^{(1)}$.

N	10 %	5 %	2·5 %	1 %
3	1·4065	1·4123	1·4137	1·4142
4	1·6454	1·6887	1·7103	1·7234
5	1·791	1·869	1·917	1·955
6	1·895	1·997	2·067	2·130
7	1·973	2·093	2·182	2·265
8	2·041	2·170	2·274	2·374
9	2·099	2·237	2·348	2·464
10	2·144	2·295	2·413	2·540
11	2·190	2·343	2·472	2·606
12	—	2·388	2·521	2·663
13	—	2·425	2·567	2·713
14	—	2·463	2·598	2·759
15	—	—	2·636	2·800
16	—	—	2·670	2·837
17	—	—	—	2·871
18	—	—	—	2·903
19	—	—	—	2·932

N.B. The same limits, with negative sign, will apply to the τ calculated from the lowest observation, i.e. to $\tau^{(N)}$.

In concluding this paper we should like it to be clear that we consider Dr W. R. Thompson has suggested a useful, practical criterion which the experi-

* This was done by backward interpolation in Table XXV of *Tables for Statisticians and Biometricians*, Part II, for $N = 5$ to 19. For the case $N = 4$ the law of equation (3) gives a rectangular distribution and for $N = 3$ a simple result follows on noting that if we write $\tau = \sqrt{2}\cos\theta$, θ is distributed uniformly between 0 and π.

menter may employ provided that he recognises that it may be inefficient in the presence of more than one outlying observation. The criterion possesses a great advantage which so many criteria that have been invented before lack, in that it provides complete control over the risk of rejecting the hypothesis tested when it is true. Our purpose has been to show that even when this control is assured, other difficulties exist which appear to be inevitably inherent in the problem of the rejection of outlying observations.

See F. E. Grubbs, Ann. Math. Stat. 21:27 (1950) and W. J. Dixon, ibid 21:488 (1950)

APPENDIX.

By J. M. C. Scott.

The object of this note is to find the values of $\tau_1, \ldots \tau_N$ which satisfy the following relations:

$$\begin{cases} |\tau_1| \geqslant |\tau_2| \geqslant \ldots \geqslant |\tau_i| \ldots \geqslant |\tau_N| & \ldots\ldots(17), \\ \tau_1+\tau_2+\ldots+\tau_N=0 & \ldots\ldots(18), \\ \tau_1^2+\tau_2^2+\ldots+\tau_N^2=N & \ldots\ldots(19), \\ |\tau_i| \text{ to be as great as possible} & \ldots\ldots(20). \end{cases}$$

Line (20) may be replaced by the requirement

$$\frac{\tau_1^2+\tau_2^2+\ldots+\tau_N^2}{\tau_i^2} \text{ to be as small as possible} \ldots\ldots(21).$$

Now (17), (18) and (21) are homogeneous, and their solutions are the solutions of (17), (18), (19) and (21)—i.e. of (17), (18), (19) and (20)—multiplied by an arbitrary factor. If this arbitrary factor is determined by

$$\tau_i=1 \ldots\ldots(22),$$

instead of by (19), we are led to consider the following problem which is equivalent to the original one:

$$\begin{cases} |\alpha_1| \geqslant |\alpha_2| \geqslant \ldots \geqslant |\alpha_i| \ldots \geqslant |\alpha_N| & \ldots\ldots(23), \\ \alpha_1+\alpha_2+\ldots+\alpha_N=0 & \ldots\ldots(24), \\ \alpha_i=1 & \ldots\ldots(25), \\ \alpha_1^2+\alpha_2^2+\ldots+\alpha_N^2 \text{ to be as small as possible} & \ldots\ldots(26). \end{cases}$$

We have to find i numbers (α_1 to α_i) numerically $\geqslant 1$, and $N-i$ numbers (α_{i+1} to α_N) numerically $\leqslant 1$, satisfying (24), (25) and (26). The order within these sets can be settled afterwards by (23), and is irrelevant at present.

Lemma 1. *The numbers α_{i+1} to α_N are equal; the positive numbers among α_1 to α_i are equal; and the negative numbers among α_1 to α_i are equal.*

Proof. If in one of these sets there are two unequal numbers, replace them by their mean. This gives a permissible set of α's but diminishes $\Sigma(\alpha^2)$, contradicting (26).

Further, the positive numbers among α_1 to α_i must be $=1$, since $\alpha_i=1$. Thus we have

$$\begin{cases} r & \text{numbers equal to } A \leqslant -1, \text{ say,} \\ i-r & \text{,,} \quad \text{,,} \quad \text{,,} +1, \\ N-i & \text{,,} \quad \text{,,} \quad \text{,,} \ C, \text{ where } |C| \leqslant 1. \end{cases}$$

Henceforward we will suppose $i<N$.

Lemma 2. $A=-1$, *if* $r>0$.

Proof. Suppose the lemma is not true; then there is an $\alpha_k<-1$. Change α_k from A to $A+\epsilon$ where $\epsilon>0$, and change α_N from C to $C-\epsilon$. This is possible if $C>-1$. It decreases $\Sigma(\alpha^2)$ by

$$A^2+C^2-(A+\epsilon)^2-(C-\epsilon)^2=2\epsilon(C-A-\epsilon)>0 \text{ for small enough } \epsilon,$$

which contradicts (26). This argument fails if $C=-1$. In this case, change the signs of α_1 to

a_{i-1}, and change a_{i+1} to a_N from -1 to $1-\frac{2}{N-i}$. This gives a permissible set of a's. If now $i<N-1$, $\Sigma(a^2)$ is reduced, contrary to (26); if $i=N-1$, $\Sigma(a^2)$ is left unchanged and some of the a's are equal to $-A>1$, which is contrary to Lemma 1. Thus Lemma 2 is established.

It follows that either $A=-1$ or $r=0$. In each case

$$C=\frac{2r-i}{N-i} \qquad \text{.....(27),}$$

leading to

$$\Sigma(a^2)=i+\frac{(2r-i)^2}{N-i} \qquad \text{.....(28).}$$

We have now to determine r so that this expression is a minimum, and to remember that

$$\tau_s^2=a_s^2\frac{\Sigma(\tau^2)}{\Sigma(a^2)}=\frac{a_s^2N}{\Sigma(a^2)} \qquad \text{.....(29).}$$

Case 1. If i is even, $r=\frac{1}{2}i$, $C=0$ and therefore

$$\Sigma(a^2)=i.$$

It follows that

$$\text{Maximum } |\tau_i|=\sqrt{\frac{N}{i}} \qquad \text{.....(30),}$$

and that of $\tau_1, \ldots \tau_i$ half are equal to $\sqrt{N/i}$ and half equal to $-\sqrt{N/i}$, while the remaining $N-i$ values are zero. This result is otherwise obvious, for if $\tau_i^2>N/i$, then

$$\sum_{s=1}^{N}(\tau_s^2)\geqslant\sum_{s=1}^{i}(\tau_s^2)>N.$$

Case 2. If i is odd, then $r=\frac{1}{2}(i\pm1)$, $C=\pm1/(N-i)$ and

$$\Sigma(a^2)=i+\frac{1}{N-i}.$$

It follows that

$$\text{Maximum } |\tau_i|=\sqrt{\frac{N}{i+\frac{1}{N-i}}} \qquad \text{.....(31).}$$

Of $\tau_1, \ldots \tau_i$,

$$\begin{cases}\frac{1}{2}(i\pm1) \text{ values}=-\sqrt{\dfrac{N}{i+\frac{1}{N-i}}},\\ \frac{1}{2}(i\mp1) \text{ values}=+\sqrt{\dfrac{N}{i+\frac{1}{N-i}}},\end{cases}$$

while

$$\tau_{i+1}=\ldots=\tau_N=\pm\sqrt{\frac{N}{(N-i)(Ni-i^2+1)}}.$$

It finally remains to consider the position where $i=N$. If N is even, the result is clearly the same as for case 1 above. If N is odd, $\frac{1}{2}(N-1)$ of the a's will equal $-(N+1)/(N-1)$ and $\frac{1}{2}(N+1)$ will equal $+1$. Therefore

$$\Sigma(a^2)=\frac{N(N+1)}{N-1}.$$

Hence

$\frac{1}{2}(N-1)$ of the τ's will equal $\pm\sqrt{(N+1)/(N-1)}$ and $\frac{1}{2}(N+1)$ will equal $\mp\sqrt{(N-1)/(N+1)}$;

thus

$$\text{Maximum } |\tau_i|=\sqrt{\frac{N-1}{N+1}} \qquad \text{.....(32).}$$

A further result used in the main paper is that of the maximum value of $\tau^{(2)}$, the second τ in order of magnitude. Following a similar line of proof, it can be readily shown that this occurs when

$$\left.\begin{aligned}\tau^{(1)}&=\tau^{(2)}=\sqrt{\tfrac{1}{2}(N-2)}\\ \tau^{(3)}&=\ldots=\tau^{(N)}=-\sqrt{2/(N-2)}\end{aligned}\right\} \qquad \text{.....(33).}$$

RELATIONS BETWEEN TWO SETS OF VARIATES*.

By HAROLD HOTELLING, Columbia University.

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1. *The Correlation of Vectors. The Most Predictable Criterion and the Tetrad Difference.* Concepts of correlation and regression may be applied not only to ordinary one-dimensional variates but also to variates of two or more dimensions. Marksmen side by side firing simultaneous shots at targets, so that the deviations are in part due to independent individual errors and in part to common causes such as wind, provide a familiar introduction to the theory of correlation; but only the correlation of the horizontal components is ordinarily discussed, whereas the complex consisting of horizontal and vertical deviations may be even more interesting. The wind at two places may be compared, using both components of the velocity in each place. A fluctuating vector is thus matched at each moment with another fluctuating vector. The study of individual differences in mental and physical traits calls for a detailed study of the relations between sets of correlated variates. For example the scores on a number of mental tests may be compared with physical measurements on the same persons. The questions then arise of determining the number and nature of the independent relations of mind and body shown by these data to exist, and of extracting from the multiplicity of correlations in the system suitable characterizations of these independent relations. As another

* Presented before the American Mathematical Society and the Institute of Mathematical Statisticians at Ann Arbor, September 12, 1935.

example, the inheritance of intelligence in rats might be studied by applying not one but s different mental tests to N mothers and to a daughter of each. Then $\frac{s(s-1)}{2}$ correlation coefficients could be determined, taking each of the mother-daughter pairs as one of the N cases. From these it would be possible to obtain a clearer knowledge as to just what components of mental ability are inherited than could be obtained from any single test.

Much attention has been given to the effects of the crops of various agricultural commodities on their respective prices, with a view to obtaining demand curves. The standard errors associated with such attempts, when calculated, have usually been found quite excessive. One reason for this unfortunate outcome has been the large portion of the variance of each commodity price attributable to crops of other commodities. Thus the consumption of wheat may be related as much to the prices of potatoes, rye, and barley as to that of wheat. The like is true of supply functions. It therefore seems appropriate that studies of demand and supply should be made by groups rather than by single commodities*. This is all the more important in view of the discovery that demand and supply curves provide altogether inadequate foundation for the discussion of related commodities, the ignoring of the effects of which has led to the acceptance as part of classical theory of results which are wrong not only quantitatively but qualitatively. It is logically as well as empirically necessary to replace the classical one-commodity type of analysis, relating for example to the incidence of taxation, utility, and consumers' surplus, by a simultaneous treatment of a multiplicity of commodities†.

The relations between two sets of variates with which we shall be concerned are those that remain invariant under internal linear transformations of each set separately. Such invariants are not affected by rotations of axes in the study of wind or of hits on a target, or by replacing mental test scores by an equal number of independently weighted sums of them for comparison with physical measurements. If measurements such as height to shoulder and difference in height of shoulder and top of head are replaced by shoulder height and stature, the invariant relations with other sets of variates will not be affected. In economics there are important linear transformations corresponding for example to the mixing of different grades of wheat in varying proportions‡. Both prices and quantities are then transformed linearly.

In this case, besides the invariants to be discussed in this paper, there will be others resulting from the fact that the transformation of quantities is not independent of that of the prices, but is contragredient to it. (Cf. Section 16 below.)

* The only published study known to the writer of groups of commodities for which standard errors were calculated is the paper of Henry Schultz, "Interrelations of Demand," in *Journal of Political Economy*, Vol. XLI. pp. 468—512, August, 1933. Some at least of the coefficients obtained are significant.

† Harold Hotelling, "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions" in *Journal of Political Economy*, Vol. XL. pp. 577—616, October, 1932, and "Demand Functions with Limited Budgets" in *Econometrica*, Vol. III. pp. 66—78, January, 1935.

‡ Harold Hotelling, "Spaces of Statistics and their Metrization" in *Science*, Vol. LXVII. pp. 149—150, February 10, 1928.

Sets of variates, which may also be regarded as many-dimensional variates, or as vectors possessed of frequency distributions, have been investigated from several different standpoints. The work of Gauss on least squares and that of Bravais, Galton, Pearson, Yule and others on multivariate distributions and multiple correlation are early examples. In "The Generalization of Student's Ratio*," the writer has given a method of testing in a manner invariant under linear transformations, and with full statistical efficiency, the significance of sets of means, of regression coefficien , and of differences of means or regression coefficients. A procedure generalizing the analysis of variance to vectors has been applied to the study of the internal structure of cells by means of Brownian movements, for which the vectors representing displacements in consecutive fifteen-second intervals were compared with their resultants to demonstrate the presence of invisible obstructions restricting the movement†. Finally, S. S. Wilks has published important work on relations among two or more sets of variates which are invariant under internal linear transformations‡. Denoting by A, B and D respectively the determinants of sample correlations within a set of s variates, within a set of t variates, and in the set consisting of all these $s+t$ variates, the distribution of the statistic,

$$z = \frac{D}{AB} \qquad \text{.....(1·1),}$$

was determined exactly by Wilks under the hypothesis that the distribution is normal, with no population correlation between any variate in one set and any in the other. Wilks also found distributions of analogous functions of three or more sets, and of other related statistics.

The statistic (1·1) is invariant under internal linear transformations of either set, as will be proved in Section 4. Another example of such a statistic is provided by the maximum multiple correlation with either set of a linear function of the other set, which has been the subject of a brief study§. This problem of finding not only a best predictor among the linear functions of one set, but at the same time the function of the other set which it predicts most accurately, will be solved in Section 3 in a more symmetrical manner. When the influence of these two linear functions is eliminated by partial correlation, the process may be repeated with the residuals. In this way we may obtain a sequence of pairs of variates, and of correlations between them, which in the aggregate will fully characterize the invariant relations between the sets, in so far as these can be represented by correlation coefficients. They will be called *canonical variates* and *canonical correlations*. Every invariant under general linear internal transformations, such for example as z, will be seen to be a function of the canonical correlations.

* *Annals of Mathematical Statistics*, Vol. II. pp. 360—378, August, 1931.

† L. G. M. Baas-Becking, Henriette van de Sande Bakhuyzen, and Harold Hotelling, "The Physical State of Protoplasm" in *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam*, Second Section, Vol. v. (1928).

‡ "Certain Generalizations in the Analysis of Variance" in *Biometrika*, Vol. XXIV. pp. 471—494, November, 1932.

§ Harold Hotelling, "The Most Predictable Criterion" in *Journal of Educational Psychology*, Vol. XXVI. pp. 139—142, February, 1935.

Observations of the values taken in N cases by the components of two vectors constitute two matrices, each of N columns. If each vector has s components, then each matrix has s rows. In this case we may consider the correlation coefficient between the C_s^N-rowed determinants in one matrix and the corresponding determinants in the other. Since a linear transformation of the variates in either set effects a linear transformation of the rows of the matrix of observations, which merely multiplies all these determinants by the same constant, it is evident that the correlation coefficient thus calculated is invariant in absolute value. We shall call it the *vector correlation* or *vector correlation coefficient*, and denote it by q. When $s = 2$, if we call the variates of one set x_1, x_2, and those of the other x_3, x_4, and r_{ij} the correlation of x_i with x_j, then it is easy to deduce with the help of the theorems stated in Section 2 below that

$$q = \frac{r_{13}r_{24} - r_{14}r_{23}}{\sqrt{(1 - r_{12}^2)(1 - r_{34}^2)}} \quad \text{.....(1·2).}$$

For larger values of s, q will have as its numerator the determinant of correlations of variates in one set with variates in the other, and as its denominator the geometric mean of the two determinants of internal correlations. A generalization of q for sets with unequal numbers of components will be given in Section 4.

Corresponding to the correlation coefficient r between two simple variates, T. L. Kelley has defined the *alienation coefficient* as $\sqrt{1 - r^2}$. The square of the correlation coefficient between x and y is the fraction of the variance of y attributable to x, while the square of the alienation coefficient is the fraction independent of x. If we adopt this apportionment of variance as a basis of generalization, we shall be consistent in calling $\sqrt{z}$ the *vector alienation coefficient*.

If there exists a linear function of one set equal to a linear function of the other—if for example x_1 is identically equal to x_3—the expression (1·2) for q reduces to a partial correlation coefficient. If one set consists of a single variate and the other of two or more, the vector correlation coincides with the multiple correlation. If each set contains only one variate, q is the simple correlation between the two. Thus the vector correlation coefficient provides a generalization of several familiar concepts.

The numerator of (1·2), known as the tetrad difference or tetrad, has been of much concern to psychologists. The vanishing in the population of all the tetrads among a set of tests is a necessary condition for the theory, propounded by Spearman, that scores on the tests are made up of a component common in varying degrees to all of them, and of independent components specific to each. The vanishing of some but not all of the tetrads is a condition for certain variants of the situation*. The sampling errors of the tetrad have therefore received much attention. In dealing with them it has been thought necessary to ignore at least three types of error:

(1) The standard error formulae used are only asymptotically valid for very large samples, with no means of determining how large a sample is necessary.

* Truman L. Kelley, *Crossroads in the Mind of Man*, Stanford University Press, 1928. This book, in addition to relevant test data and discussion, contains references to the extensive literature, a standard error formula for the tetrad, and other mathematical developments.

(2) The assumption is made implicitly that the distribution of the tetrad is normal, though this cannot possibly be the case, since the range is finite*.

(3) Since the standard error formulae involve unknown population values, these are in practice replaced by sample values. No limit is known for the errors committed in this way.

Now it is evident that to test whether the population value of the tetrad is zero—the only value of interest—is the same thing as to test the vanishing of any multiple of the tetrad by a finite non-vanishing quantity. Wishart† considered the tetrad of covariances, which is simply the product of the tetrad of correlations by the four standard deviations. For this function he found exact values of the mean and standard error, thus eliminating the first source of error mentioned above.

The exact distribution of q found in Section 8 below may be used to test the vanishing of the tetrad, eliminating the first and second sources of error. Unfortunately even this distribution involves a parameter of the population, one of the canonical correlations, which must usually be estimated from the sample, introducing again an error of the third type. However there may be cases in which this one parameter will be known from theory or from a larger sample.

Now it will be shown that q is the product of the canonical correlations. Hence at least one of these correlations is zero if the tetrad is. Thus still another test of the same hypothesis may be made in this way. Now we shall obtain for a canonical correlation vanishing in the population the extremely simple standard error formula $\frac{1}{\sqrt{n}}$, involving no unknown parameter. Thus this test evades errors of the third kind, but is subject to those of the first two, although the second is somewhat mitigated by an ultimate approach to normality, since the canonical correlations satisfy the criterion for approach to normality derived by Doob in the article cited. Further research is needed to find an exact test involving no unknown parameter. The question of whether this is possible raises a very fundamental problem in the theory of statistical inference. We shall, however, find exact distributions appropriate for testing a variety of hypotheses.

2. *Theorems on Determinants and Matrices.* We shall have frequent occasion to refer to the following well-known theorem, the proofs of which parallel those of the multiplication theorem for determinants, and which might advantageously be used in expounding many parts of the theory of statistics:

Given two arrays, each composed of m *rows and* n *columns* ($m \leqslant n$). *The determinant formed by multiplying the rows of one array by those of the other equals the*

* The first proof that the distribution of the tetrad approaches normality for large samples was given by J. L. Doob in an article, "The Limiting Distributions of Certain Statistics," in the *Annals of Mathematical Statistics*, Vol. VI. pp. 160—169 (September, 1935). The proof is applicable only if the population value of z is different from unity, i.e. if the sets x_1, x_2 and x_3, x_4 are not completely independent. If they are completely independent, the limiting distribution is of the form $\frac{1}{2}ce^{-c|t|}dt$, as Doob showed. What the distribution of the tetrad is for any finite number of cases no one knows.

† "Sampling Errors in the Theory of Two Factors" in *British Journal of Psychology*, Vol. XIX. pp. 180—187 (1928).

sum of the products of the m*-rowed determinants in the first array by the corresponding* m*-rowed determinants in the second.*

When the two arrays are identical, we have the corollary that *the symmetrical determinant of the products of rows by rows of an array of* m *rows and* n *columns* ($m \leqslant n$) *equals the sum of the squares of the* m*-rowed determinants in the array, and is therefore not negative.*

3. *Canonical Variates and Canonical Correlations. Applications to Algebra and Geometry.* If $x_1, x_2, \ldots$ are variates having zero expectations and finite covariances, we denote these covariances by

$$\sigma_{\alpha\beta} = Ex_\alpha x_\beta,$$

where E stands for the mathematical expectation of the quantity following. If new variates $x_1', x_2', \ldots$ are introduced as linear functions of the old, such that

$$x_\gamma' = \sum_\alpha c_{\gamma\alpha} x_\alpha,$$

then the covariances of the new variates are expressed in terms of those of the old by the equations

$$\sigma_{\gamma\delta}' = \sum\sum_{\alpha\beta} c_{\gamma\alpha} c_{\delta\beta} \sigma_{\alpha\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3{\cdot}1),$$

obtained by substituting the equations above directly in the definition

$$\sigma_{\gamma\delta}' = Ex_\gamma' x_\delta',$$

and taking the expectation term by term.

Now (3·1) gives also the formula for the transformation of the coefficients of a quadratic form $\sum\sum\sigma_{\alpha\beta}x_\alpha x_\beta$ when the variables are subjected to a linear transformation. Hence the problem of standardizing the covariances among a set of variates by linear transformations is algebraically equivalent to the canonical reduction of a quadratic form. The transformation of a quadratic form into a sum of squares corresponds to replacing a set of variates by uncorrelated components. It is to be observed that the fundamental nature of covariances implies that $\sum\sum\sigma_{\alpha\beta}x_\alpha x_\beta$ is a positive definite quadratic form, and that only real transformations are relevant to statistical problems.

Considering two sets of variates $x_1, \ldots, x_s$ and $x_{s+1}, \ldots, x_{s+t}$, we shall denote the covariances, in the sense of expectations of products, by $\sigma_{\alpha\beta}$, $\sigma_{\alpha i}$, and σ_{ij}, using Greek subscripts for the indices $1, 2, \ldots, s$ and Latin subscripts for $s+1, \ldots, s+t$. Determination of invariant relations between the two sets by means of the correlations or covariances among the $s+t$ variates is associated with the algebraic problem, which appears to be new, of determining the invariants of the system consisting of two positive definite quadratic forms

$$\sum\sum_{\alpha\beta}\sigma_{\alpha\beta}x_\alpha x_\beta, \quad \sum\sum_{ij}\sigma_{ij}x_i x_j,$$

in two separate sets of variables, and of a bilinear form

$$\sum\sum_{\alpha i}\sigma_{\alpha i}x_\alpha x_i$$

in both sets, under real linear non-singular transformations of the two sets separately.

Sample covariances are also transformed by the formula (3·1). The ensuing analysis might therefore equally well be carried out for a sample instead of for the population. Correlations might be used instead of covariances, either for the sample or for the population, by introducing appropriate factors, or by assuming the standard deviations to be unity.

We shall assume that there is no fixed linear relation among the variates, so that the determinant of their covariances or correlations is not zero. This implies that there is no fixed linear relation among any subset of them; consequently every *principal* minor of the determinant of $s+t$ rows is different from zero.

If we consider a function u of the variates in the first set and a function v of those in the second, such that

$$u = \sum_{\alpha} a_\alpha x_\alpha, \quad v = \sum_{i} b_i x_i,$$

the conditions
$$\Sigma\Sigma\sigma_{\alpha\beta} a_\alpha a_\beta = 1, \quad \Sigma\Sigma\sigma_{ij} b_i b_j = 1 \qquad (3{\cdot}2)$$

are equivalent to requiring the standard deviations of u and v to be unity. The correlation of u with v is then

$$R = \sum_{\alpha i} \Sigma\Sigma\sigma_{\alpha i} a_\alpha b_i \qquad (3{\cdot}3).$$

If u and v are chosen so that this correlation is a maximum, the coefficients a_α and b_i will satisfy the equations obtained by differentiating

$$\Sigma\Sigma\sigma_{\alpha i} a_\alpha b_i - \tfrac{1}{2}\lambda\Sigma\Sigma\sigma_{\alpha\beta} a_\alpha a_\beta - \tfrac{1}{2}\mu\Sigma\Sigma\sigma_{ij} b_i b_j,$$

namely
$$\sum_{i} \sigma_{\alpha i} b_i - \lambda \sum_{\beta} \sigma_{\alpha\beta} a_\beta = 0 \qquad (3{\cdot}4),$$

$$\sum_{\alpha} \sigma_{\alpha i} a_\alpha - \mu \sum_{j} \sigma_{ij} b_j = 0 \qquad (3{\cdot}5).$$

Here λ and μ are Lagrange multipliers. Their interpretation will be evident upon multiplying (3·4) by a_α and summing with respect to α, then multiplying (3·5) by b_i and summing with respect to i. With (3·2) and (3·3), this process gives

$$\lambda = \mu = R.$$

The $s+t$ homogeneous linear equations (3·4) and (3·5) in the $s+t$ unknowns a_α and b_i will determine variates u and v making R a maximum, a minimum, or otherwise stationary, if their determinant vanishes. Since $\lambda = \mu$, this condition is

$$\begin{vmatrix} -\lambda\sigma_{11} \cdots -\lambda\sigma_{1s} & \sigma_{1,s+1} \cdots \sigma_{1,s+t} \\ \cdots\cdots & \cdots\cdots \\ -\lambda\sigma_{s1} \cdots -\lambda\sigma_{ss} & \sigma_{s,s+1} \cdots \sigma_{s,s+t} \\ \sigma_{s+1,1} \cdots \sigma_{s+1,s} & -\lambda\sigma_{s+1,s+1} \cdots -\lambda\sigma_{s+1,s+t} \\ \cdots\cdots & \cdots\cdots \\ \sigma_{s+t,1} \cdots \sigma_{s+t,s} & -\lambda\sigma_{s+t,s+1} \cdots -\lambda\sigma_{s+t,s+t} \end{vmatrix} = 0 \qquad (3{\cdot}6).$$

This symmetrical determinant is the discriminant of a quadratic form $\phi - \lambda\psi$, here

$$\phi = 2\sum_{ai}\sum\sigma_{ai}z_a z_i, \quad \psi = \sum_{\alpha\beta}\sum\sigma_{\alpha\beta}z_\alpha z_\beta + \sum_{ij}\sum\sigma_{ij}z_i z_j.$$

Here ψ is positive definite because it is the sum of two positive definite quadratic forms. Consequently* all the roots of (3·6) are real. Moreover the elementary divisors are all of the first degree†. This means that the matrix of the determinant in (3·6) is reducible, by transformations which do not affect either its rank or its linear factors, to a matrix having zeros everywhere except in the principal diagonal, while the elements in this diagonal are polynomials

$$E_1(\lambda), \quad E_2(\lambda), \quad \ldots, \quad E_{s+t}(\lambda),$$

none of which contains any linear factor of the form $\lambda - \rho$ raised to a degree higher than the first. Therefore, if a simple root of (3·6) is substituted for λ, the rank is $s+t-1$; but substitution of a root of multiplicity m for λ makes the rank $s+t-m$. Consequently if a simple root is substituted for λ and μ in (3·4) and (3·5) these equations will determine values of $a_1, a_2, \ldots, a_s, b_{s+1}, \ldots, b_{s+t}$, uniquely except for constant factors whose absolute values are determinable from (3·2). Not all these quantities are zero; from this fact, and the form of (3·4) and (3·5), it is evident that at least one a_α and at least one b_i differ from zero, provided the value put for λ is not zero. The variates u and v will then be fully determinate except that they may be replaced by the pair $-u, -v$. But for a root of multiplicity m there will be m linearly independent solutions instead of one in a complete set of solutions. From these may be obtained m different pairs of variates u and v.

The coefficient of the highest power of λ in (3·6) is the product of two principal minors, both of which differ from zero because the variates have been assumed algebraically independent. The equation is therefore of degree $s+t$. We assume as a mere matter of notation, if $s \neq t$, that $s < t$. Then of the $s+t$ roots at least $t-s$ vanish; for the coefficients of λ^{t-s-1} and lower powers of λ are sums of principal minors of $2s+1$ or more rows, in which λ is replaced by zero, and every such minor vanishes, as can be seen by a Laplace expansion. Also, the sign of λ may be changed in (3·6) without changing the equation, for this may be accomplished by multiplying each of the first s rows and last t columns by -1. Therefore the negative of every root is also a root. The $s+t-(t-s) = 2s$ roots that do not necessarily vanish consist therefore of s positive or zero roots $\rho_1, \rho_2, \ldots, \rho_s$, and of the negatives of these roots. These s roots which are positive or zero we shall call the *canonical correlations* between the sets of variates; the corresponding linear functions u, v whose coefficients satisfy (3·2), (3·4) and (3·5) we call *canonical variates*‡. It is clear that every canonical correlation is the correlation coefficient between a pair of canonical variates. Hence no canonical correlation can exceed unity. The greatest canonical correlation is the maximum multiple correlation

* Maxime Bôcher, *Introduction to Higher Algebra*, New York, 1931, p. 170, Theorem 1.

† Bôcher, p. 305, Theorem 4; p. 267, Theorem 2; p. 271, Definition 3.

‡ The word "canonical" is used in the algebraic theory of invariants with a meaning consistent with that of this paper.

with either set of a disposable linear function of the other set. If u, v are canonical variates corresponding to ρ_γ, then the pair $u, -v$ or $v, -u$ is associated with the root $-\rho_\gamma$.

If a pair of canonical variates corresponding to a root ρ_γ is

$$u_\gamma = \sum_\alpha a_{\alpha\gamma} x_\alpha, \quad v_\gamma = \sum_i b_{i\gamma} x_i \qquad \text{.....(3·7),}$$

the coefficients must satisfy (3·4) and (3·5), so that

$$\sum_i \sigma_{\alpha i} b_{i\gamma} = \rho_\gamma \sum_\beta \sigma_{\alpha\beta} a_{\beta\gamma} \qquad \text{.....(3·8),}$$

$$\sum_\alpha \sigma_{\alpha i} a_{\alpha\gamma} = \rho_\gamma \sum_j \sigma_{ij} b_{j\gamma} \qquad \text{.....(3·9).}$$

Also let

$$u_\delta = \sum_\beta a_{\beta\delta} x_\beta, \quad v_\delta = \sum_i b_{i\delta} x_i \qquad \text{.....(3·10)}$$

be canonical variates associated with a canonical correlation ρ_δ. Among the four variates (3·7) and (3·10) there are six correlations. Apart from ρ_γ and ρ_δ these are obviously

$$\left.\begin{aligned} Eu_\gamma u_\delta &= \Sigma\Sigma \sigma_{\alpha\beta} a_{\beta\gamma} a_{\alpha\delta}, & Eu_\gamma v_\delta &= \Sigma\Sigma \sigma_{\alpha i} a_{\alpha\gamma} b_{i\delta} \\ Ev_\gamma u_\delta &= \Sigma\Sigma \sigma_{\alpha i} b_{i\gamma} a_{\alpha\delta}, & Ev_\gamma v_\delta &= \Sigma\Sigma \sigma_{ij} b_{i\gamma} b_{j\delta} \end{aligned}\right\} \qquad \text{.....(3·11).}$$

We shall prove that the last four are all zero. Multiply (3·8) by $a_{\alpha\delta}$ and sum with respect to α. The result, with the help of (3·11), may be written

$$Ev_\gamma u_\delta = \rho_\gamma Eu_\gamma u_\delta \qquad \text{.....(3·12).}$$

Multiplying (3·9) by $b_{i\delta}$ and summing with respect to i, we get

$$Eu_\gamma v_\delta = \rho_\gamma Ev_\gamma v_\delta \qquad \text{.....(3·13).}$$

Interchanging γ and δ in this and then using (3·12), we obtain

$$\rho_\gamma Eu_\gamma u_\delta = \rho_\delta Ev_\gamma v_\delta \qquad \text{.....(3·14).}$$

Again interchanging γ and δ, we have

$$\rho_\delta Eu_\gamma u_\delta = \rho_\gamma Ev_\gamma v_\delta.$$

If $\rho_\gamma^2 \neq \rho_\delta^2$, the last two equations show that $Eu_\gamma u_\delta = Ev_\gamma v_\delta = 0$. Hence, by (3·12) and (3·13), $Ev_\gamma u_\delta$ and $Eu_\gamma v_\delta$ vanish. Thus all the correlations among canonical variates are zero except those between the canonical variates associated with the same canonical correlation.

If ρ_s is a root of multiplicity m, it is possible by well-known processes to obtain m solutions of the linear equations such that, if

$$\begin{aligned} &a_{1\gamma}, \quad \ldots, \quad a_{s\gamma}, \quad b_{s+1,\gamma}, \quad \ldots, \quad b_{s+t,\gamma}, \\ &a_{1\delta}, \quad \ldots, \quad a_{s\delta}, \quad b_{s+1,\delta}, \quad \ldots, \quad b_{s+t,\delta}, \end{aligned}$$

are any two of these solutions, they will satisfy the orthogonality condition

$$\sum_\alpha a_{\alpha\gamma} a_{\alpha\delta} + \sum_i b_{i\gamma} b_{i\delta} = 0 \qquad \text{.....(3·15).}$$

There is no loss of generality in supposing that each of the original variates was

uncorrelated with the others in the same set and had unit variance. In this case (3·15) is equivalent to

$$Eu_\gamma u_\delta + Ev_\gamma v_\delta = 0,$$

where u_γ, v_γ, u_δ, v_δ are given by (3·7) and (3·10). For this case of equal roots we have also from (3·14),

$$\rho_\alpha (Eu_\gamma u_\delta - Ev_\gamma v_\delta) = 0.$$

If $\rho_\alpha \neq 0$, the last two equations show that $Eu_\gamma u_\delta = Ev_\gamma v_\delta = 0$, and then from (3·12) and (3·13) we have that $Ev_\gamma u_\delta = Eu_\gamma v_\delta = 0$. These correlations also vanish if $\rho_\alpha = 0$, for then the right-hand members of (3·8) and (3·9) vanish, leaving two distinct sets of equations in disjunct sets of unknowns. The solutions may therefore be chosen so that the two sums in (3·15) vanish separately.

A double zero root determines uniquely, if $s = t$, a pair of canonical variates. If $s < t$, such a root determines a canonical variate for the less numerous set, and leaves $t - s$ degrees of freedom for the choice of the other.

The reduction of our sets of variates to canonical form may be completed by the choice of new variates $v_{s+1}, v_{s+2}, \ldots, v_t$ as linear functions of the second and more numerous set (unless the numbers in the two sets are equal), uncorrelated with each other and with the canonical variates v_γ previously determined, and having unit variance. This may be done in infinitely many ways, as is well known. These variates will also be uncorrelated with the canonical variates u_γ. Indeed, if

$$v_k = \Sigma b_{jk} x_j$$

is one of them, its correlation with u_γ is, by (3·7) and (3·9),

$$Eu_\gamma v_k = \Sigma\Sigma \sigma_{\alpha j} a_{\alpha\gamma} b_{jk} = \rho_\gamma \Sigma\Sigma \sigma_{ij} b_{i\gamma} b_{jk} = \rho_\gamma Ev_\gamma v_k,$$

which vanishes because v_k was defined to be uncorrelated with v_γ.

The normal form of two sets of variates under internal linear transformations is thus found to consist of linear functions $u_1, u_2, \ldots, u_s$ of one set, and $v_1, v_2, \ldots, v_t$ of the other, such that all the correlations among these linear functions are zero, except that the correlation of u_γ with v_γ is a positive number ρ_γ ($\gamma = 1, 2, \ldots, s$). Therefore *the only invariants of the system under internal linear transformations are* $\rho_1, \rho_2, \ldots, \rho_s$, *and functions of these quantities.*

The solution of the algebraic problem mentioned at the beginning of this section, by steps exactly parallel to those just taken with the statistical problem, is the following:

The positive definite quadratic forms $\Sigma\Sigma\sigma_{\alpha\beta} x_\alpha x_\beta$, *and* $\Sigma\Sigma\sigma_{ij} x_i x_j$, *and the bilinear form* $\Sigma\Sigma\sigma_{\alpha i} x_\alpha x_i$ *with real coefficients, where the Latin subscripts are summed from* 1 *to* s *and the Greek subscripts from* $s+1$ *to* $s+t$, *and* $s \leqslant t$, *may be reduced by a real linear transformation of* $x_1, \ldots, x_s$ *and a real linear transformation of* $x_{s+1}, \ldots, x_{s+t}$ *simultaneously to the respective forms* $x_1^2 + \ldots + x_s^2$, $x_{s+1}^2 + \ldots + x_{s+t}^2$, *and* $\rho_1 x_1 x_{s+1} + \rho_2 x_2 x_{s+2} + \ldots + \rho_s x_s x_{2s}$. *A fundamental system of invariants under such transformations consists of* $\rho_1, \ldots, \rho_s$.

This algebraic theorem holds also if the quadratic forms are not restricted to be positive definite, provided (3·6) has no multiple roots and the forms are non-singular.

The normalization process we have defined may also be carried out for a sample, yielding canonical correlations $r_1, r_2, \ldots, r_s$, which may be regarded as estimates of $\rho_1, \rho_2, \ldots, \rho_s$, and associated canonical variates. With sampling problems raised in this way we shall largely be concerned in the remainder of this paper.

A further application is to geometry. In a space of N dimensions a sample of N values of a variate may be represented by a point whose coordinates are the observed values. The sample correlation between two variates is the cosine of the angle between lines drawn from the origin to the representing points, with the proviso, since deviations from means are used in the expression for a correlation, that the sum of all the coordinates of each point be zero. A sample of $s+t$ variates determines a flat space of s and one of t dimensions, intersecting at the origin, and containing the points representing the two sets of variates. In typical cases these two flat spaces do not intersect except at this one point. A complete set of metrical invariants of a pair of flat spaces is easily seen from the foregoing analysis to consist of s angles whose cosines are $r_1, \ldots, r_s$. Indeed, like all correlations, they are invariant under rotations of the N-space about the origin, and they do not depend on the particular points used to define the two flat spaces. Each of these invariants is the angle between a line in one flat space and a line in the other. One of the invariants is the *minimum* angle of this kind, and the others are in a sense stationary. The condition that the two flat spaces intersect in a line is that one of the invariant quantities $r_1, \ldots, r_s$ be unity. They intersect in a plane if two of these quantities equal unity. For two planes through a point in space of four or more dimensions, there will be two invariants r_1, r_2, of which one is the cosine of the minimum angle. If $r_1 = r_2$, the planes are *isocline*. Every line in each plane then makes the minimum angle with some line in the other. If $r_1 = r_2 = 0$, the planes are completely perpendicular; every line in one plane is then perpendicular to every line in the other. If one of these invariants is zero and the other is not, the planes are semi-perpendicular; every line in each plane is perpendicular to a certain line in the other.

The determinant of the correlations among canonical variates is

$$\Delta = \begin{vmatrix} 1 & 0 & \ldots & 0 & \rho_1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \rho_2 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & \rho_s & \ldots & 0 \\ \rho_1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \rho_2 & \ldots & 0 & 0 & 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \rho_s & 0 & 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 \end{vmatrix}$$

$$= (1-\rho_1^2)(1-\rho_2^2)\ldots(1-\rho_s^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3{\cdot}16).$$

The rank of the matrix

$$\left\| \begin{matrix} \rho_{1,s+1} & \dots\dots & \rho_{1,s+t} \\ \dots\dots & \dots\dots & \dots\dots \\ \rho_{s,s+1} & \dots\dots & \rho_{s,s+t} \end{matrix} \right\|$$

of correlations between the two sets is invariant under non-singular linear transformations of either set. Transformation to canonical variates reduces this matrix to

$$\left\| \begin{matrix} \rho_1 & 0 & \dots & 0 & \dots\dots & 0 \\ 0 & \rho_2 & \dots & 0 & \dots\dots & 0 \\ \dots\dots & \dots\dots & \dots & \dots\dots & \dots\dots & \dots\dots \\ 0 & 0 & \dots & \rho_s & \dots\dots & 0 \end{matrix} \right\|.$$

The rank is therefore the number of canonical correlations that do not vanish. This is the number of independent components common to the two sets. In the parlance of mental testing, the number of "common factors" of two sets of tests (e.g. mental and physical, or mathematical and linguistic tests) is the number of non-vanishing canonical correlations.

4. *Vector Correlation and Alienation Coefficients.* In terms of the covariances among the variates in the two sets $x_1, \dots, x_s$ and $x_{s+1}, \dots, x_{s+t}$, we define the following determinants, maintaining the convention that Greek subscripts take values from 1 to s, and Latin subscripts take values from $s+1$ to $s+t$. It will be assumed throughout that $s \leqslant t$. A is the determinant of the covariances among the variates in the first set, arranged in order: that is, the element in the αth row and βth column of A is $\sigma_{\alpha\beta}$. B is the determinant of the covariances among variates in the second set, likewise ordered. D is the determinant of $s+t$ rows containing in order all the covariances among all the variates of both sets. C is obtained from D by replacing the covariances among the variates of the first set, including their variances, by zeros. Symbolically,

$$A = |\sigma_{\alpha\beta}|, \quad B = |\sigma_{ij}|, \quad C = \left| \begin{array}{c|c} 0 & \sigma_{\alpha i} \\ \hline \sigma_{i\alpha} & \sigma_{ij} \end{array} \right|, \quad D = \left| \begin{array}{c|c} \sigma_{\alpha\beta} & \sigma_{\alpha i} \\ \hline \sigma_{i\alpha} & \sigma_{ij} \end{array} \right|.$$

Suppose now that new variates $x_1', \dots, x_s'$ are defined in terms of the old variates in the first set by the s equations

$$x_\gamma' = \Sigma c_{\gamma\alpha} x_\alpha.$$

The new covariances are then expressed in terms of the old by (3·1). The determinant of these new covariances, which we shall denote by A', may by (3·1) and the multiplication theorem of determinants be expressed as the product of three determinants, of which two equal the determinant $c = |c_{\gamma\alpha}|$ of the coefficients of the transformation, while the third is A. If the variates of the second set are subjected to a transformation of determinant d, the determinants of covariances among the new variates analogous to those defined above are readily seen in this way to equal

$$A' = c^2 A, \quad B' = d^2 B, \quad C' = c^2 d^2 C, \quad D' = c^2 d^2 D \quad \dots\dots\dots\dots (4·1).$$

Thus A, B, C, D are *relative invariants* under internal transformations of the two sets of variates.

The ratios $$Q^2 = \frac{(-1)^s C}{AB} \text{ and } Z = \frac{D}{AB} \quad \text{.....(4·2)}$$

we shall call respectively the squares of the *vector correlation coefficient* or *vector correlation*, and of the *vector alienation coefficient*. It is evident that both are absolute invariants under internal transformations of the two sets, since their values computed from transformed variates have numerators and denominators multiplied by the same factor c^2d^2, in accordance with (4·1).

The notation just used is appropriate to a population, but the same definitions and reasoning may be applied to a sample. We denote by q^2 and z the same functions of the sample covariances that Q^2 and Z, respectively, have been defined to be of the population covariances.

A particularly simple linear transformation consists of dividing each variate by its standard deviation. The covariances among the new variates are then the same as their correlations, which are also the correlations among the old variates. Hence, in the definitions of the vector correlation and alienation coefficients, the covariances may be replaced by the correlations. For example, if $s = t = 2$, the squared vector correlation in a sample may be written

$$q^2 = \frac{\begin{vmatrix} 0 & 0 & r_{13} & r_{14} \\ 0 & 0 & r_{23} & r_{24} \\ r_{31} & r_{32} & 1 & r_{34} \\ r_{41} & r_{42} & r_{43} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_{12} \\ r_{12} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & r_{34} \\ r_{34} & 1 \end{vmatrix}} = \frac{(r_{13}r_{24} - r_{14}r_{23})^2}{(1 - r_{12}^2)(1 - r_{34}^2)} \quad \text{.....(4·3).}$$

The vector correlation coefficient will always be taken as the positive square root of q^2 or of Q^2 (which are seen below to be positive) when $s < t$, and usually also when $s = t$. However, if in accordance with (4·3) we write

$$q = \frac{r_{13}r_{24} - r_{14}r_{23}}{\sqrt{(1 - r_{12}^2)(1 - r_{34}^2)}} \quad \text{.....(4·4),}$$

it is evident that q may be positive for some samples of a particular set of variates, and negative for other samples. It may sometimes be advantageous, as in testing whether two samples arose from the same population, to retain the sign of q for each sample, since this provides evidence in addition to that given by the absolute value of q. But unless otherwise stated we shall always regard q as the positive root of q^2. Likewise, Q, $\sqrt{z}$ and $\sqrt{Z}$ will denote the positive roots unless otherwise specifically indicated in each case. A transformation of either set will reverse the sign of the algebraic expression (4·4) if the determinant of the transformation is negative. This will be true of a simple interchange of two variates; for example, $x_1' = x_2$, $x_2' = x_1$ has the determinant -1. On the other hand, the sign is conserved if the determinant of the transformation is positive. Such considerations apply whenever $s = t$.

Since the vector correlation and alienation coefficients are invariants, they may be computed on the assumption that the variates are canonical. In this case $A = B = 1$, and D is given by (3·16). To obtain C we replace the first s 1's in the principal diagonal of (3·16) by 0's. It then follows that

$$C = (-1)^s \rho_1^2 \rho_2^2 \dots \rho_s^2.$$

This confirms that the value of Q^2 given in (4·2) is positive. In this way the vector correlation and alienation coefficients are expressible in terms of the canonical correlations by the equations

$$Q = \pm \rho_1 \rho_2 \dots \rho_s, \quad Z = (1-\rho_1^2)(1-\rho_2^2)\dots(1-\rho_s^2) \quad \dots\dots\dots(4{\cdot}5),$$

$$q = \pm r_1 r_2 \dots r_s, \quad z = (1-r_1^2)(1-r_2^2)\dots(1-r_s^2) \quad \dots\dots\dots(4{\cdot}6).$$

From these results it is obvious that both the vector correlation and vector alienation coefficients are confined to values not exceeding unity. Also Z and z are necessarily positive, except that they vanish if, and only if, all the variates in one set are linear functions of those in the other.

Since the denominator of (4·4) is obviously less than unity, and since we have just shown that $q \leqslant 1$, the tetrad must be still less. This simple proof that the tetrad is between -1 and $+1$ shows the falsity of the idea that the range of the tetrad is from -2 to $+2$, which has gained some currency. An equivalent proof in vector notation was communicated to the writer by E. B. Wilson.

The only case in which Z can attain its maximum value unity is that in which all the canonical correlations vanish. In this case no variate in either set is correlated with any variate in the other, so that the two sets are completely independent, at least if the distribution is normal. Moreover, $Q = 0$. On the other hand, the only case in which Q can be unity is that in which all the canonical correlations are unity. In this event, $Z = 0$; also, the variates in the first set are linear functions of those in the second. Thus either z, $1-q$, or $1-q^2$ might be used as an index of independence, while we might use q, q^2 or $1-z$ as a measure of relationship between the two sets. The work of Wilks alluded to in Section 1 provides an exact distribution of z on the hypothesis of complete independence, a distribution which may thus be used to test this hypothesis.

If we regard the elements of A, B and C as sample covariances, we have in case $s = t$ a simple interpretation of q. Consider the two matrices of observations on the two sets of variates in N individuals, in which each row corresponds to a variate and each column to an individual observed. From Section 2 it is evident that the square of the sum of the products of corresponding s-rowed determinants in the two matrices is $(-1)^s N^{2s} C$; also that the sums of squares of the s-rowed determinants in the two matrices are $N^s A$ and $N^s B$. Therefore q is simply the product-moment correlation coefficient between corresponding s-rowed determinants.

The generalized variance of a set of variates may be defined as the determinant of their ordered covariances, such as A or B. Let $\xi_1, \xi_2, \dots, \xi_s$ be estimates

respectively of $x_1, x_2, \ldots, x_s$ obtained from $x_{s+1}, \ldots, x_{s+t}$ by least squares, and let the regression equations be

$$\xi_a = \sum_i b_{ai} x_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4{\cdot}7).$$

The appropriateness of Q as a generalization of the correlation coefficient, and of $\sqrt{Z}$ as a generalization of the alienation coefficient, will be apparent from the following theorem:

The ratio of the generalized variance of $\xi_1, \ldots, \xi_s$ to that of $x_1, \ldots, x_s$ is Q^2. The ratio of the generalized variance of $x_1 - \xi_1, x_2 - \xi_2, \ldots, x_s - \xi_s$ to that of $x_1, \ldots, x_s$ is Z.

This theorem is expressed in terms of the population, but an exactly parallel one holds for a sample.

Proof: If $x_1, \ldots, x_s$ be subjected to a linear transformation of determinant c, and if $\xi_1, \ldots, \xi_s$ be subjected to the same transformation (i.e. a transformation with the same coefficients), then $x_1 - \xi_1, \ldots, x_s - \xi_s$ will also be subjected to this transformation. The generalized variances of all three of these sets of variates will be multiplied by the same constant c^2, just as in (4·1) we found that $A' = c^2 A$. Ratios among the three determinants will therefore be absolutely invariant; consequently our theorem is true if it is true when the original variates are canonical. Suppose, then, that this is the case. Since each canonical variate is correlated with only one of the other set, the regression equations (4·7) reduce simply to

$$\xi_a = \rho_a x_{a+s} \quad (a = 1, \ldots, s).$$

Since the variance of x_{a+s} is unity, that of ξ_a is ρ_a^2; that of the deviation $x_a - \xi_a$ is $1 - \rho_a^2$. Since the canonical variates $x_1, \ldots, x_s$ are mutally uncorrelated, the same is true of the ξ_a, and also of the $x_a - \xi_a$. The generalized variance of the canonical variates is unity; that of the ξ_a is the product of the elements in its principal diagonal, namely $\rho_1^2 \rho_2^2 \ldots \rho_s^2$; and the generalized variance of the $x_a - \xi_a$ is $(1 - \rho_1^2)\ldots(1 - \rho_s^2)$. In view of (4·5), this proves the theorem.

A further property of the vector correlation is obvious from the final paragraph of Section 3:

A necessary and sufficient condition that the number of components in an uncorrelated set of components common to two sets of variates be less than the number of variates in either set is that the vector correlation be zero.

When $s = 2$ the canonical correlations not only determine the vector correlation and alienation coefficients but are determined by them. If as usual we take q positive, (4·6) becomes $q = r_1 r_2$, $z = (1 - r_1^2)(1 - r_2^2)$, whence

$$r_1^2 + r_2^2 = 1 - z + q^2, \quad r_1^2 r_2^2 = q^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4{\cdot}8).$$

Solving, and denoting the greater canonical correlation by r_1, we have

$$\left.\begin{aligned} r_1 &= \tfrac{1}{2}\left[\sqrt{(1+q)^2 - z} + \sqrt{(1-q)^2 - z}\right] \\ r_2 &= \tfrac{1}{2}\left[\sqrt{(1+q)^2 - z} - \sqrt{(1-q)^2 - z}\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4{\cdot}9).$$

Since the canonical correlations are real, $(r_1 - r_2)^2$ is positive; therefore

$$z \leqslant (1-q)^2 \quad \text{.....(4·10).}$$

In like manner, the vector correlation and alienation coefficients in the population are subject not only to the inequalities $0 \leqslant Q^2 \leqslant 1$, $0 \leqslant Z \leqslant 1$, but also, when $s = 2$, to

$$Z \leqslant (1-Q)^2.$$

These inequalities become equalities when the roots are equal.

The fundamental equation (3·6), regarded as an equation in λ^2, has as roots the squares of the canonical correlations. Hence, by (4·8), it reduces it to the form

$$\lambda^4 - (1 - z + q^2)\lambda^2 + q^2 = 0 \quad \text{.....(4·11),}$$

where $s = 2$.

5. *Standard Errors.* The canonical correlations and the coefficients of the canonical variates are defined in Section 3 in such a way that they are continuous functions of the covariances, with continuous derivatives of all orders, except for certain sets of values corresponding to multiple or zero roots, within the domain of variation for which the covariances are the coefficients of a positive definite quadratic form. This is true for the canonical reduction of a sample as well as for that of a population. The probability that a random sample from a continuous distribution will yield multiple roots is zero; and sample covariances must always be the coefficients of a positive definite form.

We shall in this section derive asymptotic standard errors, variances and covariances for the canonical correlations on the assumption that those in the population are unequal, and that the population has the multiple normal distribution. From these we shall derive standard errors for the vector correlation and alienation coefficients q and z. The deviation of sample from population values in these as in most cases have variances of order n^{-1}, and distributions approaching normality of form as n increases*.

Let $x_1, \ldots, x_p$ be a normally distributed set of variates of zero means and covariances

$$\sigma_{ij} = Ex_i x_j \quad \text{.....(5·1).}$$

For a sample of N in which x_{if} is the value of x_i observed in the fth individual, the sample covariance of x_i and x_j is

$$s_{ij} = \frac{\sum_f (x_{if} - \bar{x}_i)(x_{jf} - \bar{x}_j)}{N-1} = \frac{\sum x_{if} x_{jf} - N\bar{x}_i \bar{x}_j}{N-1} \quad \text{.....(5·2),}$$

where $\bar{x}_i$ and $\bar{x}_j$ are the sample means. To simplify the later work, we introduce the *pseudo-observations*, x_{if}', defined in terms of the observations by the equations

$$x_{if}' = \sum_{g=1}^{N} c_{fg} x_{ig} \quad \text{.....(5·3),}$$

* For a proof of approach to normality for a general class of statistics including those with which we deal, cf. Doob, *op. cit.*

where the quantities c_{fg}, independent of i and therefore the same for all the variates x_i, are the coefficients of an orthogonal transformation, such that

$$c_{N1} = c_{N2} = \ldots = c_{NN} = \frac{1}{\sqrt{N}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}4).$$

Since the transformation is orthogonal we must have

$$\sum_h c_{fh} c_{gh} = \delta_{fg} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}5),$$

where δ_{fg} is the Kronecker delta, equal to unity if $f=g$, but to zero if $f \neq g$. The coefficients c_{fg} may be chosen in an infinite variety of ways consistently with these requirements, but will be held fixed throughout the discussion. Since linear functions of normally distributed variates are normally distributed, the pseudo-observations are normally distributed. Their population means are, from (5·3),

$$Ex_{if}' = \Sigma c_{fg} Ex_{ig} = 0,$$

since the original variates were assumed to have zero means. Also, since the expectation of the product of independent variates is zero, and since the different individuals in a sample are assumed independent, so that, by (5·1),

$$Ex_{ih} x_{jk} = \delta_{hk} \sigma_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}6),$$

we have, from (5·3), (5·6) and (5·5),

$$\left.\begin{aligned} Ex_{if}' x_{jg}' &= \sum_{hk}\sum c_{fh} c_{gk} Ex_{ih} x_{jk} \\ &= \sum_{hk}\sum c_{fh} c_{gk} \delta_{hk} \sigma_{ij} \\ &= \sum_h c_{fh} c_{gh} \sigma_{ij} \\ &= \delta_{fg} \sigma_{ij} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}7).$$

From (5·4) and (5·3) it is clear that

$$x_{iN}' = \frac{\sum_g x_{ig}}{\sqrt{N}} = \sqrt{N}\, \bar{x}_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}8).$$

The equations (5·3) may, on account of their orthogonality, be solved in the form

$$x_{if} = \pm \Sigma c_{gf} x_{ig}'.$$

Therefore, by (5·5),

$$\sum_f x_{if} x_{jf} = \sum\sum\sum c_{gf} c_{hf} x_{ig}' x_{jh}' = \sum\sum \delta_{gh} x_{ig}' x_{jh}' = \sum_g x_{ig}' x_{jg}'.$$

Substituting this result and (5·8) in (5·2), we find that the final term of the sum cancels out. Introducing therefore the symbol S for summation from 1 to $N-1$ with respect to the second subscript, and putting also

$$n = N - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}9),$$

we have the compact result

$$s_{ij} = \frac{S x_{ig}' x_{jg}'}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}10).$$

Since the pseudo-observations are normally distributed with the covariances (5·7) and zero means, they have exactly the same distribution as the observations in a random sample of n from the original population. The equivalence of the mean product (5·10) with the sample covariance (5·2) establishes the important principle that *the distribution of covariances in a sample of $n+1$ is exactly the same as the distribution of mean products in a sample of n,* if the parent population is normally distributed about zero means. Use of this principle will considerably simplify the discussions of sampling.

An important extension of this consideration lies in the use of deviations, not merely from sample means, but from regression equations based on other variates. In such cases the number of degrees of freedom n to be used is the difference between the sample number and the number of constants in each of the regression equations, which number must be the same for all the deviations. The estimate of covariance of deviations in the ith and jth variates to be used is then the sum of the products of corresponding deviations, divided by n. This may also be regarded as the mean product of the values of x_i and x_j in n pseudo-observations, as above, without elimination of the means or of the extraneous variates. The sampling distributions with which we shall be concerned will all be expressed in terms of the number of degrees of freedom n, rather than in terms of the number of observations N. This will permit immediately of the extension, which is equivalent to replacing all the correlations, in terms of which our statistics may be defined, by partial correlations representing the elimination of a particular set of variates, the same in all cases.

A variance is of course the covariance of a variate with itself, so that this whole discussion of covariances is equally applicable to variances.

The characteristic function of a multiple normal distribution with zero means is well known to be

$$M(t_1, t_2, \dots) = Ee^{\Sigma t_i x_i} = e^{\frac{1}{2}\Sigma\Sigma \sigma_{ij} t_i t_j}.$$

The moments of the distribution are the derivatives of the characteristic function, evaluated for $t_1 = t_2 = \dots = 0$. From the fourth derivative with respect to t_i, t_j, t_k and t_m it is easy to show in this way that

$$Ex_i x_j x_k x_m = \sigma_{ij}\sigma_{km} + \sigma_{im}\sigma_{jk} + \sigma_{ik}\sigma_{jm} \quad \dots\dots\dots\dots\dots\dots (5{\cdot}11).$$

From (5·10) we have

$$Es_{ij}s_{km} = \frac{1}{n^2} SSEx_{ig}' x_{jg}' x_{kf}' x_{mf}' \quad \dots\dots\dots\dots\dots\dots (5{\cdot}12).$$

Now if $f \neq g$,

$$Ex_{ig}' x_{jg}' x_{kf}' x_{mf}' = (Ex_{ig}' x_{jg}')(Ex_{kf}' x_{mf}') = \sigma_{ij}\sigma_{km} \dots\dots\dots\dots\dots\dots (5{\cdot}13),$$

since the expectation of the product of *independent* quantities is the product of their expectations. Of the n^2 terms in the double sum in (5·12), $n^2 - n$ are equal to (5·13). The remaining n terms are those for which $f = g$, and each of them equals (5·11). Hence

$$Es_{ij}s_{km} = \sigma_{ij}\sigma_{km} + \frac{1}{n}(\sigma_{im}\sigma_{jk} + \sigma_{ik}\sigma_{jm}).$$

Inasmuch as

$$Es_{ij} = \sigma_{ij},$$

we have, if we put $$d\sigma_{ij} = s_{ij} - \sigma_{ij},$$

for the deviation of sample from population value, that the sampling covariance of two covariances is

$$E d\sigma_{ij} d\sigma_{km} = E s_{ij} s_{km} - \sigma_{ij}\sigma_{km},$$

whence $$E d\sigma_{ij} d\sigma_{km} = \frac{1}{n}(\sigma_{ik}\sigma_{jm} + \sigma_{im}\sigma_{jk}) \quad \text{.....(5·14).}$$

This is a fundamental formula from which may be derived directly a number of more familiar special cases. For example, to obtain the variance of a variance, merely put $i = j = k = m$, which gives

$$\sigma_{s_{ii}}^2 = E(d\sigma_{ii})^2 = \frac{2\sigma_{ii}^2}{n}.$$

Returning from these general considerations to the problem of canonical correlations, we recall from (3·2) and (3·3) that for any particular canonical correlation ρ_1,

$$\Sigma\Sigma\sigma_{\alpha\beta}a_\alpha a_\beta = 1, \quad \Sigma\Sigma\sigma_{ij}b_i b_j = 1, \quad \rho_1 = \Sigma\Sigma\sigma_{\alpha i}a_\alpha b_i \text{.....(5·15),}$$

where α and β are summed from 1 to s, and i and j from $s+1$ to $s+t$. Any particular set of sampling errors $d\sigma_{AB}$ in the covariances determines a corresponding set of sampling errors in the a_α and b_i and in ρ_1, for these quantities are definite analytic functions of the covariances except when ρ_1 is a multiple or zero root of (3·6), cases which we now exclude from consideration. In terms of the derivatives of these functions we define

$$da_\alpha = \Sigma\Sigma\frac{\partial a_\alpha}{\partial\sigma_{AB}}d\sigma_{AB}, \quad db_i = \Sigma\Sigma\frac{\partial b_i}{\partial\sigma_{AB}}d\sigma_{AB}, \quad d\rho_1 = \Sigma\Sigma\frac{\partial\rho_1}{\partial\sigma_{AB}}d\sigma_{AB} \quad (5·16),$$

where $d\sigma_{AB} = s_{AB} - \sigma_{AB}$, and the summations are over all values of A and B from 1 to $s+t$. Then differentiating (5·15) we have

$$\left.\begin{aligned} &\Sigma\Sigma(2\sigma_{\alpha\beta}a_\alpha da_\beta + a_\alpha a_\beta d\sigma_{\alpha\beta}) = 0, \quad \Sigma\Sigma(2\sigma_{ij}b_i db_j + b_i b_j d\sigma_{ij}) = 0, \\ &d\rho_1 = \Sigma\Sigma(\sigma_{\alpha i}a_\alpha db_i + \sigma_{\alpha i}b_i da_\alpha + a_\alpha b_i d\sigma_{\alpha i}) \end{aligned}\right\} \quad (5·17).$$

Let us now suppose that the variates are in the population canonical. This assumption does not entail any loss of generality as regards ρ_1, since ρ_1 is an invariant under transformations of the variates of either set. Since a_α is the coefficient of x_α in the expression for one of the canonical variates, which we take to be x_1, we have in the population $a_1 = 1$, $a_2 = a_3 = \ldots = a_s = 0$. In the same way,

$$b_{s+1} = 1, \quad b_{s+2} = \ldots = b_{s+t} = 0.$$

Also, since the covariances among canonical variates are the elements of the determinant in (3·16), we have

$$\sigma_{\alpha\beta} = \delta_{\alpha\beta}, \quad \sigma_{ij} = \delta_{ij}, \quad \sigma_{\alpha i} = \delta_{\alpha+s,\,i}\rho_\alpha \text{.....(5·18),}$$

the Kronecker deltas being equal to unity if the two subscripts are equal, and otherwise vanishing. When these special values of the a's, b's and σ's are substituted in (5·17) most of the terms drop out, leaving the simple equations

$$\left.\begin{aligned} &2da_1 + d\sigma_{11} = 0, \quad 2db_{s+1} + d\sigma_{s+1,\,s+1} = 0, \\ &d\rho_1 = \rho_1 db_{s+1} + \rho_1 da_1 + d\sigma_{1,\,s+1} \end{aligned}\right\} \text{.....(5·19).}$$

Substituting from the first two in the third of these equations, we get

$$d\rho_1 = d\sigma_{1,s+1} - \tfrac{1}{2}\rho_1(d\sigma_{11} + d\sigma_{s+1,s+1}) \quad \text{.....(5·20).}$$

For any other simple root ρ_2 we have in the same way

$$d\rho_2 = d\sigma_{2,s+2} - \tfrac{1}{2}\rho_2(d\sigma_{22} + d\sigma_{s+2,s+2}) \quad \text{.....(5·21).}$$

Squaring (5·20), taking the expectation, using the fundamental formula (5·14), and finally substituting the canonical values (5·18), we have

$$\left.\begin{aligned} nE(d\rho_1)^2 &= \sigma_{11}\sigma_{s+1,s+1} + \sigma_{1,s+1}{}^2 - \rho_1(2\sigma_{11}\sigma_{1,s+1} + 2\sigma_{s+1,s+1}\sigma_{1,s+1}) \\ &\quad + \tfrac{1}{4}\rho_1{}^2(2\sigma_{11}{}^2 + 4\sigma_{1,s+1}{}^2 + 2\sigma_{s+1,s+1}{}^2) \\ &= 1 + \rho_1{}^2 - \rho_1(2\rho_1 + 2\rho_1) + \tfrac{1}{4}\rho_1{}^2(2 + 4\rho_1{}^2 + 2) \\ &= (1 - \rho_1{}^2)^2 \end{aligned}\right\} \quad \text{(5·22).}$$

Treating the product of (5·20) and (5·21) in the same way we obtain

$$Ed\rho_1 d\rho_2 = 0 \quad \text{.....(5·23).}$$

A sample canonical correlation r_1 may be expanded about ρ_1 in a Taylor series of the form

$$r_1 = \rho_1 + \Sigma\Sigma \frac{\partial\rho_1}{\partial\sigma_{AB}} d\sigma_{AB} + \tfrac{1}{2}\Sigma\Sigma\Sigma\Sigma \frac{\partial^2\rho_1}{\partial\sigma_{AB}\partial\sigma_{CD}} d\sigma_{AB}\, d\sigma_{CD} + \dots \quad \text{...(5·24),}$$

or, by the last of (5·16),

$$r_1 - \rho_1 = d\rho_1 + \dots \quad \text{.....(5·25).}$$

The expectation of the product of any number of the sampling deviations $d\sigma_{AB}$ is a fixed function of the σ's divided by a power of n whose exponent increases with the number of the quantities $d\sigma_{AB}$ in the product. Since $Ed\sigma_{AB} = 0$, we have from (5·24) and (5·14) that $E(r_1 - \rho_1)$ is of order n^{-1}. Hence squaring (5·25) and using (5·22), we find that the sampling variance of r_1 is given by $\dfrac{(1-\rho_1{}^2)^2}{n}$, apart from terms of higher order in n^{-1}. If by the standard error of r_1 we understand the leading term in the asymptotic expansion of the square root of the variance, we have for this standard error

$$\sigma_{r_1} = \frac{1 - \rho_1{}^2}{\sqrt{n}} \quad \text{.....(5·26).}$$

It is remarkable that this standard error of a canonical correlation is of exactly the same form as that of a product-moment correlation coefficient calculated directly from data, at least so far as the leading term is concerned.

The covariance of two statistics or their correlation would ordinarily be of order n^{-1}; but from (5·23) it appears that the covariance of r_1 and r_2 is of order n^{-2} at least. All these results hold as between any pair of simple non-vanishing roots. To summarize:

Let $\rho_1, \rho_2, \dots, \rho_p$ be any set of simple non-vanishing roots of (3·6). *For sufficiently large samples these will be approximated by certain of the canonical correlations $r_1, r_2, \dots, r_p$ of the samples in such a way that, when $r_\gamma - \rho_\gamma$ is divided by the standard error*

$$\sigma_{r_\gamma} = \frac{1 - \rho_\gamma{}^2}{\sqrt{n}} \quad (\gamma = 1, 2, \dots, p) \dots \quad \text{.....(5·27),}$$

the resulting variates have a distribution which, as n increases, approaches the normal distribution of p independent variates of zero means and unit standard deviations.

For small samples there will be ambiguities as to which root of the determinantal equation for the sample is to be regarded as approximating a particular canonical correlation of the population. As n increases, the sample roots will separately cluster more and more definitely about individual population roots.

If a canonical correlation ρ_γ is zero, and if $s=t$, the foregoing result is applicable with the qualification that sample values r_γ approximating ρ_γ must not all be taken positive, but must be assigned positive and negative values with equal probabilities. Alternatively, if we insist on taking all the sample canonical correlations as positive, the distribution will be that of absolute values of a normally distributed variate.

To prove this, suppose that the determinantal equation has zero as a double root. For sample covariances sufficiently near those in the population, there will be a root r close to zero, which will be very near the value of λ obtained by dropping from the equation all but the term in λ^2 and that independent of λ. The latter is for $s=t$ a perfect square, and the former does not vanish, since the zero root is only a double one. Hence r is the ratio of a polynomial in the s_{AB}'s to a non-vanishing regular function in the neighbourhood. This means that the differential method applicable to non-vanishing roots is also valid here, and that, since the derivatives are continuous, (5·27) holds even when $\rho_\gamma = 0$.

Since a tetrad difference is proportional to a vector correlation, which is the product of the canonical correlations, the question whether the tetrad differs significantly from zero is equivalent to the question whether a canonical correlation is significantly different from zero. This may be tested by means of the standard error (5·27), which reduces in this case to $\frac{1}{\sqrt{n}}$. Since this is independent of unknown parameters, we have here a method of meeting the third of the difficulties mentioned in Section 1 in connection with testing the significance of the tetrad.

For $s=2$, a zero root is of multiplicity t at least. From the final result in § 9 below it may be deduced that if zero is a root of multiplicity exactly t, if r is the corresponding sample canonical correlation, and if $s=2$, then nr^2 has the χ^2 distribution with $t-1$ degrees of freedom. This provides a means of testing the significance of a sample canonical correlation in all cases in which $s=2$.

We shall conclude this section by deriving standard error formulae for the vector correlation and vector alienation coefficients, assuming the canonical correlations in the population all distinct. Differentiating (4·5) and supposing all canonical correlations positive we have

$$dQ = \Sigma \frac{d\rho_\gamma}{\rho_\gamma}, \quad dZ = -2Z\Sigma \frac{\rho_\gamma d\rho_\gamma}{1-\rho_\gamma^2}.$$

Taking the expectations of the squares and products of these expressions and using

(5·22) and (5·23), we obtain for the variances and covariance, apart from terms of higher order in n^{-1},

$$\sigma_q = Q\sqrt{\frac{1}{n}\sum_{\gamma=1}^{s}\frac{(1-\rho_\gamma^2)^2}{\rho_\gamma^2}}, \quad \sigma_z = 2Z\sqrt{\frac{\rho_1^2+\ldots+\rho_s^2}{n}},$$

$$E\,dQ\,dZ = -\frac{2}{n}QZ\Sigma(1-\rho_\gamma^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5{\cdot}28).$$

For the case $s=2$ these formulae reduce with the help of (4·5) to

$$\sigma_q = \sqrt{\frac{(1-Q^2)^2 - Z(1+Q^2)}{n}}, \quad \sigma_z = 2Z\sqrt{\frac{1-Z+Q^2}{n}},$$

$$E\,dQ\,dZ = -\frac{2}{n}QZ(1+Z-Q^2).$$

6. *Examples, and an Iterative Method of Solution.* The correlations obtained by Truman L. Kelley* among tests in (1) reading speed, (2) reading power, (3) arithmetic speed, and (4) arithmetic power are given by the elements of the following determinant, in which the rows and columns are arranged in the order given:

$$D = \begin{vmatrix} 1{\cdot}0000 & {\cdot}6328 & {\cdot}2412 & {\cdot}0586 \\ {\cdot}6328 & 1{\cdot}0000 & -{\cdot}0553 & {\cdot}0655 \\ {\cdot}2412 & -{\cdot}0553 & 1{\cdot}0000 & {\cdot}4248 \\ {\cdot}0586 & {\cdot}0655 & {\cdot}4248 & 1{\cdot}0000 \end{vmatrix} = {\cdot}4129.$$

These correlations were obtained from a sample of 140 seventh-grade school children. Let us inquire into the relations of arithmetical with reading abilities indicated by these tests.

The two-rowed minors of D in the upper left, lower right, and upper right corners are respectively

$$A = {\cdot}5996, \quad B = {\cdot}8195, \quad \sqrt{C} = {\cdot}01904.$$

Hence, by (4·2),

$$q^2 = {\cdot}0007377, \quad q = {\cdot}027161, \quad z = {\cdot}84036 \quad \ldots\ldots\ldots\ldots\ldots(6{\cdot}1).$$

By means of (4·9) or (4·11) these values give for the canonical correlations

$$r_1 = {\cdot}3945, \quad r_2 = {\cdot}0688 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6{\cdot}2).$$

In this case $n = N-1 = 139$, and the standard error (5·27) reduces, for the hypothesis of a zero canonical correlation in the population, to $\frac{1}{\sqrt{139}} = {\cdot}0848$. It is plain, therefore, that r_2 is not significant, so that we do not have any evidence here of more than one common component of reading and arithmetical abilities.

Whether we have convincing evidence of *any* common component is another question. It is tempting to compare the value of r_1 also with the standard error ·0848 for the purpose of answering this question, which would give a decidedly significant value. This however is not a sensitive procedure for testing the hypothesis

* *Op. cit.*, p. 100. These are the raw correlations, not corrected for attenuation.

that there is no common factor; for this hypothesis of complete independence would mean that both canonical correlations would in the population be zero; they would therefore be a quadruple root of the fundamental equation, to which the standard error is not applicable. Other tests for complete independence will be considered in Section 11; these have a sound basis, and one of them (discovered by Wilks) gives approximately ·0001 as the probability of a value of z as small as or smaller than the value found above. We conclude that reading and arithmetic involve one common mental factor but, so far as these data show, only one.

Linear functions $a_1x_1 + a_2x_2$ and $b_3x_3 + b_4x_4$ having maximum correlation with each other may be used either to predict arithmetical from reading ability or vice versa. The coefficients will satisfy (3·4) and (3·5); when in these equations we substitute $r_1 = ·3945$ for λ and μ, and the given correlations for the covariances, and divide by $-\lambda = -·3945$, we have

$$\begin{aligned} a_1 + ·6328a_2 - ·6114b_3 - ·1485b_4 &= 0, \\ ·6328a_1 + a_2 + ·1402b_3 - ·1660b_4 &= 0, \\ -·6114a_1 + ·1402a_2 + b_3 + ·4248b_4 &= 0, \\ -·1485a_1 - ·1660a_2 + ·4248b_3 + b_4 &= 0. \end{aligned}$$

The fourth equation must be dependent on the preceding three, so we ignore it except for a final checking. Replacing b_4 by unity we may solve the first three equations, which are symmetrical, by the usual least-square method. Thus we write the coefficients, without repetition, in the form

1·0000	·6328	−·6114	−·1485	·8729
	1·0000	·1402	−·1660	1·6070
		1·0000	·4248	·9536

the last column consisting of the sums of the elements written or understood in the respective rows. The various divisions, multiplications and subtractions involved in solving the equations are applied to the elements in the rows, including those in the check column, which at every stage gives the sum of the elements written or understood in a row. In the array above, the coefficients of each equation begin in the first row and proceed downward to the diagonal, then across to the right, and this scheme is followed with the reduced set of equations obtained by eliminating an unknown, which is done in such a way as to preserve symmetry. This process yields finally the ratios

$$a_1 : a_2 : b_3 : b_4 = -2·7772 : 2·2655 : -2·4404 : 1.$$

Therefore the linear functions of arithmetical and reading scores that predict each other most accurately are proportional to $-2·7772x_1 + 2·2655x_2$ and $-2·4404x_3 + x_4$, respectively. It is for these weighted sums that the maximum correlation ·3945 is attained.

From the same individuals, Kelley obtained the correlations in the following table, in which the first two rows correspond to the arithmetic speed and power

tests cited above, while the others are respectively memory for words, memory for meaningful symbols, and memory for meaningless symbols:

$$\left\|\begin{array}{cc:ccc} 1{\cdot}0000 & {\cdot}4248 & {\cdot}0420 & {\cdot}0215 & {\cdot}0573 \\ {\cdot}4248 & 1{\cdot}0000 & {\cdot}1487 & {\cdot}2489 & {\cdot}2843 \\ \hdashline {\cdot}0420 & {\cdot}1487 & 1{\cdot}0000 & {\cdot}6693 & {\cdot}4662 \\ {\cdot}0215 & {\cdot}2489 & {\cdot}6693 & 1{\cdot}0000 & {\cdot}6915 \\ {\cdot}0573 & {\cdot}2843 & {\cdot}4662 & {\cdot}6915 & 1{\cdot}0000 \end{array}\right\|.$$

From this we find $q^2 = {\cdot}0003209$, $q = {\cdot}01792$, $z = {\cdot}902466$,

whence $r_1 = {\cdot}3073$, $r_2 = {\cdot}0583$.

Since in this case $s \neq t$, we cannot say as before that the standard error of r_2 when $\rho = 0$ is $n^{-\frac{1}{2}} = {\cdot}0848$. But, putting $\chi^2 = nr_2^2 = {\cdot}472$, with two degrees of freedom, we find $P = {\cdot}79$, so that r_2 is far from significant. However r_1 is decidedly significant.

In view of the tests in Section 11, we conclude in this case also that there is evidence of one common component but not of two.

If each of the two sets contains more than two variates, the two invariants q and z do not suffice to determine the coefficients of the various powers of λ in the determinantal equation, so that its roots can no longer be calculated in the foregoing manner. The coefficients in the equation will involve other rational invariants in addition to q and z, but we shall not be concerned with these, and it is desirable to have a procedure that does not require their calculation, or the explicit determination and solution of the equation. It is also desirable to avoid the explicit solution of the sets of linear equations (3·4) and (3·5) when the variates are numerous, since the labour of the direct procedure then becomes excessive. These computational difficulties are analogous to those in the determination of the principal axes of a quadric in n-space, or of the principal components of a set of statistical variates, problems for which an iterative procedure has been found useful, and has been proved to converge to the correct values in all cases*. We shall now show how a process partly iterative in character may be applied to determine canonical variates and canonical correlations between two sets.

If in the s equations (3·4) we regard $\lambda a_1, \lambda a_2, \ldots, \lambda a_s$ as the unknowns, we may solve for them in terms of the b's by the methods appropriate for solving normal equations. Indeed, the matrix of the coefficients of the unknowns is symmetrical; and in the solving process it is only necessary to carry along, instead of a single column of right-hand members, t columns, from which the coefficients of $b_{s+1}, \ldots, b_{s+t}$ in the expressions for $a_1, \ldots, a_s$ are to be determined. The entries initially placed

* Harold Hotelling, "Analysis of a Complex of Statistical Variables into Principal Components" in *Journal of Educational Psychology*, Vol. xxiv. pp. 417—441 and 498—520 (September and October, 1933), Section 4.

in these columns are of course the covariances between the two sets. Let the solution of these equations consist of the s expressions

$$\lambda a_\alpha = \sum_i g_{\alpha i} b_i \quad (\alpha = 1, 2, \ldots, s) \quad \ldots\ldots\ldots\ldots (6{\cdot}3).$$

In exactly the same way the t equations (3·5), with μ replaced by λ, may be solved for $\lambda b_{s+1}, \ldots, \lambda b_{s+t}$ in the form

$$\lambda b_i = \sum_\beta h_{i\beta} a_\beta \quad (i = s+1, \ldots, s+t) \ldots\ldots\ldots (6{\cdot}4).$$

If we substitute from (6·4) in (6·3) and set

$$k_{\alpha\beta} = \sum_i g_{\alpha i} h_{i\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6{\cdot}5),$$

we have

$$\lambda^2 a_\alpha = \sum_\beta k_{\alpha\beta} a_\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6{\cdot}6).$$

Now if an arbitrarily chosen set of numbers be substituted for $a_1, \ldots, a_s$ in the right-hand members of (6·6), the sums obtained will be proportional to the numbers substituted only if they are proportional to the true values of $a_1, \ldots, a_s$. If, as will usually be the case, the proportionality does not hold, the sums obtained, multiplied or divided by any convenient constant, may be used as second approximations to solutions $a_1, \ldots, a_s$ of the equations. Substitution of these second approximations in the right-hand members of (6·6) gives third approximations which may be treated in the same way; and so on. Repetition of this process gives repeated sets of trial values, whose ratios will be seen below to approach as limits those among the true values of $a_1, \ldots, a_s$. The factor of proportionality λ^2 in (6·6) becomes r_1^2, the square of the largest canonical correlation. When the quantities $a_1', \ldots, a_s'$ eventually determined as sufficiently nearly proportional to $a_1, \ldots, a_s$ are substituted in the right-hand members of (6·4), there result quantities $b_{s+1}', \ldots, b_{s+t}'$ proportional to $b_{s+1}, \ldots, b_{s+t}$, apart from errors which may be made arbitrarily small by continuation of the iterative process. The factor of proportionality to be applied in order to obtain linear functions with unit variance is the same for the a's and the b's; from (3·2), (3·4), and (3·5) it may readily be shown that if from the quantities obtained we calculate

$$m = \frac{r_1}{\sqrt{\sum\sum \sigma_{\alpha i} a_\alpha' b_i'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6{\cdot}7),$$

then the true coefficients of the first pair of canonical variates are $ma_1', \ldots, ma_s'$, $mb_{s+1}', \ldots, mb_{s+t}'$.

In the iterative process, if $a_1, \ldots, a_s$ represent trial values at any stage, those at the next stage will be proportional to

$$a_\alpha' = \sum k_{\alpha\beta} a_\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6{\cdot}8).$$

Another application of the process gives

$$a_\gamma'' = \sum k_{\gamma\alpha} a_\alpha',$$

whence, substituting, we have

$$a_\gamma'' = \sum k_{\gamma\beta}^{(2)} a_\beta,$$

provided we put

$$k_{\gamma\beta}^{(2)} = \sum_\alpha k_{\gamma\alpha} k_{\alpha\beta}.$$

The last equation is equivalent to the statement that the matrix K^2 of the coefficients $k_{\gamma\beta}^{(2)}$ is the square of the matrix K of the $k_{\alpha\beta}$. It follows therefore that one application of the iterative process by means of the squared matrix is exactly equivalent to two successive applications with the original matrix. This means that if at the beginning we square the matrix only half the number of steps will subsequently be required for a given degree of accuracy.

The number of steps required may again be cut in half if we square K^2, for with the resulting matrix K^4 one iteration is exactly equivalent to four with the original matrix. Squaring again we obtain K^8, with which one iteration is equivalent to eight, and so on. This method of accelerating convergence is also applicable to the calculation of principal components*. It embodies the root-squaring principle of solving algebraic equations in a form specially suited to determinantal equations.

After each iteration it is advisable to divide all the trial values obtained by a particular one of them, say the first, so as to make successive values comparable. The value obtained for a_1, if this is the one used to divide the rest at each step, will approach r_1^2 if the matrix K is used in iteration, but will approach r_1^4 if K^2 is used, r_1^8 if K^4 is used, and so forth. When stationary values are reached, they may well be subjected once to iteration by means of K itself, both in order to determine r_1^2 without extracting a root of high order, and as a check on the matrix-squaring operations.

If our covariances are derived from a sample from a continuous multivariate distribution, it is infinitely improbable that the equation in ω,

$$\begin{vmatrix} k_{11}-\omega & k_{12} & \dots\dots & k_{1s} \\ k_{21} & k_{22}-\omega & \dots\dots & k_{2s} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ k_{s1} & k_{s2} & \dots\dots & k_{ss}-\omega \end{vmatrix} = 0,$$

has multiple roots. If we assume that the roots $\omega_1, \omega_2, \dots, \omega_s$ are all simple, and regard $a_1, \dots, a_s$ as the homogeneous coordinates of a point in $s-1$ dimensions which is moved by the collineation (6·8) into a point $(a_1', \dots, a_s')$, we know† that there exists in this space a transformed system of coordinates such that the collineation is represented in terms of them by

$$\bar{a}_1' = \omega_1 \bar{a}_1, \quad \bar{a}_2' = \omega_2 \bar{a}_2, \quad \dots, \quad \bar{a}_s' = \omega_s \bar{a}_s.$$

Another iteration yields a point whose transformed homogeneous coordinates are proportional to

$$\omega_1^2 \bar{a}_1, \quad \omega_2^2 \bar{a}_2, \quad \dots, \quad \omega_s^2 \bar{a}_s.$$

Continuation of this process means, if ω_1 is the root of greatest absolute value, that

* Another method of accelerated iterative calculation of principal components is given by T. L. Kelley in *Essential Traits of Mental Life*, Cambridge, Mass., 1935. A method similar to that given above is applied to principal components by the author in *Psychometrika*, Vol. 1. No. 1 (1936).

† Bôcher, p. 293.

the ratio of the first transformed coordinates to any of the others increases in geometric progression. Consequently the moving point approaches as a limit the invariant point corresponding to this greatest root. Therefore the ratios of the trial values of $a_1, \ldots, a_s$ will approach those among the coefficients in the expression for the canonical variate corresponding to the greatest canonical correlation. Thus the iterative process is seen to converge, just as in the determination of principal components.

After the greatest canonical correlation and the corresponding canonical variates are determined, it is possible to construct a new matrix of covariances of deviations from these canonical variates. When the iterative process is applied to this new matrix, the second largest canonical correlation and the corresponding canonical variates are obtained. This procedure may be carried as far as desired to obtain additional canonical correlations and variates, as in the method of principal components; but the later stages of the process will yield results which will usually be of diminishing importance. The modification of the matrix is somewhat more complicated than in the case of principal components, and we shall omit further discussion of this extension.

The process of obtaining iteratively the greatest canonical correlation, the most predictable criterion, and the best predicter may be illustrated if we imagine that, with three variates in each set, we have obtained from a sample the matrix of correlations

$$\left\|\begin{array}{ccc:ccc} 1{\cdot}0 & {\cdot}7 & {\cdot}1 & {\cdot}5 & {\cdot}4 & {\cdot}2 \\ {\cdot}7 & 1{\cdot}0 & {\cdot}1 & {\cdot}4 & {\cdot}3 & {\cdot}5 \\ {\cdot}1 & {\cdot}1 & 1{\cdot}0 & {\cdot}2 & {\cdot}2 & {\cdot}4 \\ \hdashline {\cdot}5 & {\cdot}4 & {\cdot}2 & 1{\cdot}0 & {\cdot}8 & {\cdot}6 \\ {\cdot}4 & {\cdot}3 & {\cdot}2 & {\cdot}8 & 1{\cdot}0 & {\cdot}7 \\ {\cdot}2 & {\cdot}5 & {\cdot}4 & {\cdot}6 & {\cdot}7 & 1{\cdot}0 \end{array}\right\|.$$

From the first three rows we obtain the set of normal equations indicated by

$$\begin{array}{ccccccc} 1{\cdot}0 & {\cdot}7 & {\cdot}1 & {\cdot}5 & {\cdot}4 & {\cdot}2 & 2{\cdot}9 \\ & 1{\cdot}0 & {\cdot}1 & {\cdot}4 & {\cdot}3 & {\cdot}5 & 3{\cdot}0 \\ & & 1{\cdot}0 & {\cdot}2 & {\cdot}2 & {\cdot}4 & 2{\cdot}0 \end{array}$$

Here the second and third rows are understood to be filled out with unwritten terms in such a way as to make the matrix consisting of the first three columns symmetric. The entries in the last column are the sums of those written or understood in the respective rows preceding them. By linear operations on the rows, equivalent to solving the equations, they are reduced to

$$\begin{array}{ccccccc} 1 & & & {\cdot}423 & {\cdot}362 & -{\cdot}316 & 1{\cdot}470 \\ & 1 & & {\cdot}089 & {\cdot}031 & {\cdot}685 & 1{\cdot}804 \\ & & 1 & {\cdot}149 & {\cdot}161 & {\cdot}362 & 1{\cdot}671 \end{array}$$

This is the numerical equivalent of (6·3). Hence $g_{\alpha i}$ is the element in the αth row and ith column of the matrix

$$G = \begin{Vmatrix} \cdot 423 & \cdot 362 & -\cdot 316 \\ \cdot 089 & \cdot 031 & \cdot 685 \\ \cdot 149 & \cdot 161 & \cdot 362 \end{Vmatrix}.$$

From the last three columns of the given matrix of correlations we obtain likewise the normal equations indicated by

$$\begin{array}{ccccccc} 1\cdot 0 & \cdot 8 & \cdot 6 & \cdot 5 & \cdot 4 & \cdot 2 & 3\cdot 5 \\ & 1\cdot 0 & \cdot 7 & \cdot 4 & \cdot 3 & \cdot 2 & 3\cdot 4 \\ & & 1\cdot 0 & \cdot 2 & \cdot 5 & \cdot 4 & 3\cdot 4 \end{array}$$

The three columns before the check column appear in the same order as in the lower left corner of the matrix of correlations. The solutions of these equations, corresponding to (6·4), are the elements of the matrix

$$H = \begin{Vmatrix} \cdot 522 & \cdot 385 & \cdot 054 \\ \cdot 121 & -\cdot 385 & -\cdot 199 \\ -\cdot 198 & \cdot 539 & \cdot 507 \end{Vmatrix},$$

in which $h_{i\beta}$ is the element in the ith row and βth column. Upon multiplying the rows of G by the columns of H we find that $k_{\alpha\beta}$, defined by (6·5), is in the αth row and βth column of

$$K = GH = \begin{Vmatrix} \cdot 327 & -\cdot 147 & -\cdot 209 \\ -\cdot 085 & \cdot 392 & \cdot 346 \\ \cdot 026 & \cdot 190 & \cdot 160 \end{Vmatrix}.$$

The check columns are used to verify the calculation to this point, and may be used also at the next stage, which is to compute, by multiplying the rows of K by its columns,

$$K^2 = \begin{Vmatrix} \cdot 114 & -\cdot 145 & -\cdot 153 \\ -\cdot 052 & \cdot 232 & \cdot 209 \\ -\cdot 003 & \cdot 101 & \cdot 086 \end{Vmatrix},$$

and in the same way,

$$K^4 = \begin{Vmatrix} \cdot 021 & -\cdot 066 & -\cdot 061 \\ -\cdot 019 & \cdot 082 & \cdot 074 \\ -\cdot 006 & \cdot 033 & \cdot 029 \end{Vmatrix}.$$

The iteration process may now be begun with the trial values 1, 1, 1; when this set (which may be regarded as a vector) is multiplied by the rows of K^4 there results simply the set of sums of rows, namely

$$-\cdot 106 \qquad \cdot 137 \qquad \cdot 056.$$

Dividing all three of these by the first we have

$$1\cdot 0 \qquad -1\cdot 3 \qquad -\cdot 5.$$

Multiplying this vector by the rows of K^4 gives

$$\cdot 137 \quad -\cdot 163 \quad -\cdot 063,$$

which upon division by the first becomes

$$1\cdot 00 \quad -1\cdot 19 \quad -\cdot 46.$$

Multiplication of this vector by the rows of K^4 and division by the first resulting element gives

$$1\cdot 00 \quad -1\cdot 18 \quad -\cdot 46,$$

which upon another repetition of the process recurs exactly to the two decimal places. We therefore return to the matrix K with these trial values; multiplying them by the rows of K, dividing by the first, and then repeating the process once, we have the values

$$\cdot 5968 \quad -\cdot 707 \quad -\cdot 272,$$

which, divided by the first, become

$$a_1' = 1, \quad a_2' = -1\cdot 187, \quad a_3' = -\cdot 456,$$

are stationary under further iterations, and are correct to three decimal places. The last divisor, ·5968, is the square of the greatest canonical correlation, also correct to three places; hence $r_1 = \sqrt{\cdot 5968} = \cdot 773$. Substitution of a_1', a_2', a_3', in the right members of (6·4), which comes to the same thing as multiplication by the rows of H, yields

$$b_4' = \cdot 040, \quad b_5' = \cdot 669, \quad b_6' = -1\cdot 069.$$

Then from (6·7) we have $m = 1\cdot 016$; when this is multiplied by the values of a_α' and b_i' just found, there result the coefficients in the expressions for the leading canonical variates, namely

$$u_1 = 1\cdot 016x_1 - 1\cdot 217x_2 - \cdot 463x_3,$$
$$v_1 = \cdot 041x_4 + \cdot 680x_5 - 1\cdot 086x_6,$$

which have unit variances and the correlation ·773.

7. *The Vector Correlation as a Product of Correlations or of Cosines.* We shall in this section define certain linear functions of the variates in each set, forming two sequences, of which the product of the correlations between corresponding members is the vector correlation q. This result will be used in Section 8 to obtain an exact sampling distribution of q. The resolution, though valid with respect to the population, needs for our purposes to be made with reference to a sample. We shall use the pseudo-observations defined in Section 5, but shall write the sample covariance (5·10) in the form

$$s_{LM} = \frac{Sx_L x_M}{n} \qquad \text{.....(7·1),}$$

where S stands for summation from 1 to n, the number of degrees of freedom, and where L and M stand for an arbitrary pair, equal or unequal, of the subscripts $1, 2, \ldots, s+t$.

The sequences of variates which we shall consider may be defined as follows. First, let $x_1' = x_1$. Then let x_α' ($\alpha = 2, 3, \ldots, s$) be the difference between x_α and

a least-square estimate of x_α in terms of $x_1, \ldots, x_{\alpha-1}$, all divided by such a constant that the variance of x_α' is unity. To define the other sequence, let x_{s+1}' be a linear function of $x_{s+1}, \ldots, x_{s+t}$ having maximum correlation with x_1; and let $x_{s+\beta}'$ $(\beta = 2, \ldots, s)$ be a linear function uncorrelated with $x_{s+1}', \ldots, x_{s+\beta-1}'$, and having maximum correlation with x_β'. All these are to have unit variances. For a sample we may set aside as infinitely improbable the possibility that any of these new variates should be indeterminate. Putting R_β for the correlation of x_β' with $x_{s+\beta}'$, we shall find that

$$q = R_1 R_2 \ldots R_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}2).$$

The process will be more perspicuous in geometrical than in algebraic language because of the simplicity of the geometry associated with samples from normal distributions, and the remainder of this section will be in geometrical terms. In the space of n dimensions in which the pseudo-observations of a variate are the coordinates of a point, there is for each variate a spherically symmetrical distribution of probability density centred at the origin. Let X_L denote the point whose coordinates are the pseudo-observations $x_{L1}, x_{L2}, \ldots, x_{Ln}$ on the variate x_L $(L = 1, 2, \ldots, s+t)$. Let P_t be the flat space of t dimensions containing the origin O and the points $X_{s+1}, \ldots, X_{s+t}$ determined by the second set of variates. Perpendicular to OX_1 will be a flat space of $n-1$ dimensions, whose intersection with P_t will in general be of $t-1$ dimensions. Denote this intersection by P_{t-1}. Let P_{t-2} be the flat space of $t-2$ dimensions contained in P_{t-1} and perpendicular to OX_2; and so forth.

Let X_1' be the point on OX_1 of unit distance from the origin. We further define points $X_2', \ldots, X_s'$, all at unit distance from the origin, such that $OX_1', OX_2', \ldots, OX_s'$ are mutually perpendicular, and such that OX_2' is coplanar with OX_1 and OX_2; OX_3' is in the same flat 3-space with OX_1, OX_2 and OX_3; and so forth. The coordinates

$$\begin{matrix} x_{11}' & x_{12}' & \ldots & x_{1n}' \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_{s1}' & x_{s2}' & \ldots & x_{sn}' \end{matrix}$$

of these s points will thus satisfy

$$S x_\alpha' x_\beta' = \delta_{\alpha\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}3),$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, equal to unity if $\alpha = \beta$ but otherwise zero, and where S stands for summation from 1 to n.

Let us also rotate the n axes so that the first t of them lie in P_t; and let an internal transformation be performed upon the t variates of the second set such that, for this particular sample, the coordinates representing them become

$$\begin{Vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots 1 \ldots & 0 \end{Vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}4).$$

None of these transformations affects the value of q, and we have, adapting the definition (4·2) to this case, by replacing the covariances by the functions (7·1) of $x_{11}', \ldots, x_{ss}'$ and of the elements of (7·4), and then multiplying each row of each determinant by n,

$$q^2 = (-1)^s \frac{C'}{A'B'},$$

where now $A' = B' = 1$, while

$$C' = \begin{vmatrix} 0 \ldots 0 & x_{11}' \ \ldots\ldots \ x_{1t}' \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ 0 \ldots 0 & x_{s1}' \ \ldots\ldots \ x_{st}' \\ x_{11}' \ldots x_{s1}' & 1 \quad 0 \ \ldots \ 0 \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ x_{1t}' \ldots x_{st}' & 0 \quad 0 \ \ldots \ 1 \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}5).$$

Now letting Σ stand for summation from 1 to t, we introduce determinants

$$D_\beta = \begin{vmatrix} \Sigma x_1'^2 \ \ldots\ldots \ \Sigma x_1' x_\beta' \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \Sigma x_\beta' x_1' \ \ldots\ldots \ \Sigma x_\beta'^2 \end{vmatrix} \quad (\beta = 1, 2, \ldots, s) \quad \ldots\ldots\ldots\ldots (7{\cdot}6).$$

Upon expanding (7·5) with respect to the first s rows and columns we find, with the help of Section 2,

$$q^2 = (-1)^s C' = D_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}7).$$

Now any line perpendicular to OX_1 and OX_2 is perpendicular to all the lines in the plane of these two, in particular to OX_2'. Hence P_{t-2}, which consists of lines perpendicular to OX_1 and OX_2, is perpendicular to OX_2'. In like manner, P_{t-3} is perpendicular to OX_1', OX_2' and OX_3'; and in general $P_{t-\beta}$ is perpendicular to $OX_1', OX_2', \ldots, OX_\beta'$.

Since $P_{t-\beta}$ lies entirely within P_t, the coordinates of any point U in $P_{t-\beta}$ will be linearly dependent on the rows of (7·4), and so of the form

$$u_1, u_2, \ldots, u_t, 0, 0, \ldots, 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}8).$$

The orthogonality of OU to $OX_1', \ldots, OX_\beta'$ means that

$$\Sigma u x_\alpha' = 0 \quad (\alpha = 1, 2, \ldots, \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}9).$$

Now let $\theta_{\beta+1}$ denote the angle that $OX_{\beta+1}'$ makes with $P_{t-\beta}$; that is, $\theta_{\beta+1}$ is the minimum angle of $OX_{\beta+1}'$ with a line OU such that the coordinates of U are of the form (7·8) and satisfy (7·9). Without loss of generality we may also take U at unit distance from the origin, so that

$$\Sigma u^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}10).$$

Since $Sx_{\beta+1}'^2 = 1$ by (7·3), we then have

$$\cos\theta_{\beta+1} = \Sigma u x_{\beta+1}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}11).$$

To determine the minimum angle we therefore differentiate with respect to $u_1, \ldots, u_t$ the expression

$$\Sigma u x_{\beta+1}' - \tfrac{1}{2}\gamma \Sigma u^2 - \lambda_1 \Sigma u x_1' - \ldots - \lambda_\beta \Sigma u x_\beta',$$

where $\gamma, \lambda_1, \ldots, \lambda_\beta$ are Lagrange multipliers. This gives

$$\lambda_1 x_{1h}' + \ldots + \lambda_\beta x_{\beta h}' = x_{\beta+1,h}' - \gamma u_h \quad (h = 1, \ldots, t) \quad \ldots\ldots\ldots(7{\cdot}12).$$

Multiply (7·12) by u_h and sum with respect to h. The left member disappears by (7·9), and from (7·10) and (7·11) we have

$$\gamma = \cos\theta_{\beta+1} \quad \ldots\ldots\ldots(7{\cdot}13).$$

Upon multiplying (7·12) by $x_{\alpha h}'$, summing with respect to h, and using (7·9), we have, for $\alpha = 1, 2, \ldots, \beta$,

$$\lambda_1 \Sigma x_1' x_\alpha' + \ldots + \lambda_\beta \Sigma x_\beta' x_\alpha' = \Sigma x_\alpha' x_{\beta+1}' \quad \ldots\ldots\ldots(7{\cdot}14).$$

Eliminating $\lambda_1, \ldots, \lambda_\beta$ from the $\beta+1$ equations (7·14) and (7·12) we have

$$\begin{vmatrix} \Sigma x_1'^2 & \ldots\ldots & \Sigma x_1' x_\beta' & \Sigma x_1' x_{\beta+1}' \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \Sigma x_\beta' x_1' & \ldots\ldots & \Sigma x_\beta'^2 & \Sigma x_\beta' x_{\beta+1}' \\ x_{1h}' & \ldots\ldots & x_{\beta h}' & x_{\beta+1,h}' - \gamma u_h \end{vmatrix} = 0 \quad \ldots\ldots\ldots(7{\cdot}15).$$

Multiply the last row of this determinant by $x_{\beta+1,h}'$ and sum with respect to h from 1 to t. The last element, with the help of (7·11) and (7·13), reduces to

$$\Sigma x_{\beta+1}'^2 - \gamma^2,$$

and so, from (7·6), we have

$$\gamma^2 = \frac{D_{\beta+1}}{D_\beta} \quad \ldots\ldots\ldots(7{\cdot}16).$$

Hence, from (7·13),

$$\cos\theta_{\beta+1} = \sqrt{\frac{D_{\beta+1}}{D_\beta}} \quad (\beta = 1, 2, \ldots, s-1) \quad \ldots\ldots\ldots(7{\cdot}17).$$

The cosine of the angle which OX_1 makes with P_t is

$$\cos\theta_1 = \sqrt{\Sigma x_1'^2} = \sqrt{D_1} \quad \ldots\ldots\ldots(7{\cdot}18).$$

Multiplying together all the equations (7·17) and (7·18) and recalling (7·7), we obtain

$$q = \cos\theta_1 \cos\theta_2 \ldots \cos\theta_s \quad \ldots\ldots\ldots(7{\cdot}19).$$

It is obvious that the correlations R_β defined at the beginning of this section have the property that

$$R_\beta = \cos\theta_\beta,$$

so that (7·19) is equivalent to (7·2).

8. *An Exact Sampling Distribution of q.* We shall now deduce the exact distribution of q in samples from a multivariate normal population in which the vector correlation is zero, for the case in which one of the sets consists of exactly two variates. From (4·5) it follows in this case that at least one of the two canonical correlations is zero. If the numbers of variates in both sets are 2, we have essentially the case of the tetrad difference; the distribution will then be symmetrical, since the population value is assumed to be zero. Let $\rho_2 = 0$, and for brevity put ν for ρ_1^2, which will be a parameter of the distribution.

The angle θ_1 defined in Section 7 between the line OX_1 determined by the sample values of the first variate and the flat space P_t determined by those of the second set has the property that $R_1 = \cos\theta_1$ is the multiple correlation of x_1 with the second set of variates. The population value of this multiple correlation is ρ_1. We assume all the variates subject to random sampling. In this case R_1 will have the "A" distribution discovered by R. A. Fisher*. In our notation, with samples of $n+1$ from which the means have been eliminated, or in samples of $n+k$ from which k degrees of freedom have been removed by least-squares elimination of other variates, the distribution of R_1 is

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{t}{2}\right)\Gamma\left(\frac{n-t}{2}\right)}(1-\nu)^{\frac{n}{2}}(R_1^2)^{\frac{t-2}{2}}(1-R_1^2)^{\frac{n-t-2}{2}}F\left(\frac{n}{2},\frac{n}{2},\frac{t}{2},\nu R_1^2\right)d(R_1^2) \quad \ldots\ldots(8{\cdot}1),$$

with F denoting the hypergeometric function.

The points X_2' of Section 7 corresponding to an infinity of samples form a globular cluster having spherical symmetry with centre at the origin, in the flat space of $n-1$ dimensions perpendicular to OX_1. In this flat space is the space P_{t-1}, which makes with OX_2' the angle θ_2. Hence $R_2 = \cos\theta_2$ has the distribution of a multiple correlation coefficient in samples from an *uncorrelated* normal population, with $t-1$ "independent" variates. We replace n in (8·1) by $n-1$, t by $t-1$, R_1 by R_2, ν by zero, and have

$$\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-t}{2}\right)}(R_2^2)^{\frac{t-3}{2}}(1-R_2^2)^{\frac{n-t-2}{2}}d(R_2^2) \quad \ldots\ldots\ldots(8{\cdot}2).$$

From (7·2) we have $q = R_1R_2$. Hence put $R_2 = \frac{q}{R_1}$, $dR_2 = \frac{dq}{R_1}$ in (8·2), multiply by (8·1), and integrate with respect to R_1 from q to 1. This gives the distribution of q in the form

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{t}{2}\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma^2\left(\frac{n-t}{2}\right)}(1-\nu)^{\frac{n}{2}}(q^2)^{\frac{t-3}{2}}d(q^2)$$

$$\times\int_{q^2}^1[(1-R^2)(R^2-q^2)]^{\frac{n-t-2}{2}}(R^2)^{\frac{-n+t+1}{2}}F\left(\frac{n}{2},\frac{n}{2},\frac{t}{2},\nu R^2\right)d(R^2)\ldots(8{\cdot}3),$$

where the subscript is dropped from the variable of integration. Now

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{t}{2}\right)\Gamma\left(\frac{t-1}{2}\right)} = \frac{(n-2)!}{2^{n-t}(t-2)!}.$$

* "The General Sampling Distribution of the Multiple Correlation Coefficient" in *Proceedings of the Royal Society*, Vol. CXXI, A (1928), p. 660.

Making this substitution and changing the variable of integration to

$$x = \frac{1-R^2}{1-q^2},$$

we have therefore

$$\frac{(n-2)!}{2^{n-t}(t-2)!\,\Gamma^2\left(\frac{n-t}{2}\right)}(1-\nu)^{\frac{n}{2}}(1-q^2)^{n-t-1}(q^2)^{\frac{t-3}{2}}\,d(q^2)$$

$$\times\int_0^1[x(1-x)]^{\frac{n-t-2}{2}}[1-(1-q^2)x]^{\frac{-n+t+1}{2}}F\left\{\frac{n}{2},\frac{n}{2},\frac{t}{2},\nu[1-(1-q^2)x]\right\}dx \qquad \text{.....(8·4).}$$

If it is supposed that the values of the variates in the second set are fixed, instead of varying normally from sample to sample, Fisher's distribution "C" of the multiple correlation coefficient should be used instead of his "A" distribution. This gives for q a new distribution, which for $s=2$ may be written in terms of a confluent hypergeometric function

$$\frac{\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-4}{2}\right)}e^{\frac{-\nu}{2}}(1-q^2)^{n-3}q\,dq\int_0^1[x(1-x)]^{\frac{n-4}{2}}[1-(1-q^2)x]^{\frac{-n+1}{2}}$$

$$\times F\left\{\frac{n}{2},1,\frac{1}{2},\frac{\nu q^2}{1-(1-q^2)x}\right\}dx \qquad \text{.....(8·5).}$$

However the conditions of sampling under which q is likely to be used are such that (8·4) appears to be the more important form, and we shall give no further consideration to (8·5).

By extension of the reasoning above the distribution of q may be found for larger values of s, provided all but one of the parameters $\rho_1, \ldots, \rho_s$ vanish. Thus for $s=3$ we have, from (7·2),

$$q = R_1 R_2 R_3 = q' R_3,$$

where q' has the distribution (8·3), i.e. (8·4), while R_3 has the distribution obtained from (8·2) by replacing R_2 by R_3, n by $n-1$, and t by $t-1$. Combining this with (8·4) in the same manner that (8·2) was combined with (8·1) to produce (8·3), the new distribution of q is obtained. This process may be repeated to obtain the distribution for values of s as great as desired; but it must of course be remembered that $s \leqslant t$.

It is tempting to try to obtain the general distribution of q, without our assumption that all but one of the quantities $\rho_1, \ldots, \rho_s$ are zero, by treating these as population values of the multiple correlations whose distributions are used successively in finding the distribution of q as above. However this suggested procedure appears to be incorrect. If $\rho_2 \neq 0$, the centre of the globular cluster formed by the projected X_2' points will have a centre which is not on P_{t-1}, where it should be if the multiple correlation distribution were to be amplified.

9. *Moments of q. The Distribution for Large Samples.* We shall derive the even moments of the distribution of Section 8, assuming that ν does not take

either of the extreme values 0 and 1. The latter is the case in which q becomes a partial correlation coefficient, the theory of which is well understood. The case $\nu = 0$, corresponding to complete independence, is a very simple one, concerning which all information desired may be obtained from Section 11. The moments will be obtained by processes involving repeated interchanges of order of the processes of integration, differentiation, and summation of series. It will be observed that the uniform convergence and continuity required to justify these interchanges exist, provided ν is definitely *between* 0 and 1 without taking either of these values.

The odd moments of q about zero, which require no consideration unless $s = t$, vanish in this case when only one of the canonical correlations is different from zero, since the distribution of q must then be symmetrical. Let μ_{2k} be the $2k$th moment of q, which is also the kth moment of q^2, about zero. To determine its value, multiply (8·3) by q^{2k} and integrate with respect to q^2 from 0 to 1. In the double integral thus obtained, a reversal of the order of integration means that q^2 will vary from 0 to R^2, and then R^2 will vary from 0 to 1. The first integration in this new order may be effected at once, since upon putting $q^2 = R^2 z$, $d(q^2) = R^2 dz$, we have

$$\int_0^{R^2} (q^2)^{\frac{t-3}{2}+k} (R^2 - q^2)^{\frac{n-t-2}{2}} d(q^2) = (R^2)^{\frac{n-3}{2}+k} \frac{\Gamma\left(\frac{t-1}{2}+k\right)\Gamma\left(\frac{n-t}{2}\right)}{\Gamma\left(\frac{n-1}{2}+k\right)}.$$

Therefore

$$\mu_{2k} = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{t}{2}\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-t}{2}\right)\Gamma\left(\frac{n-1}{2}+k\right)} (1-\nu)^{\frac{n}{2}}$$

$$\times \int_0^1 (R^2)^{\frac{t-2}{2}+k} (1-R^2)^{\frac{n-t-2}{2}} F\left(\frac{n}{2}, \frac{n}{2}, \frac{t}{2}, \nu R^2\right) d(R^2).$$

Expanding the hypergeometric series and integrating term by term, since

$$\int_0^1 (R^2)^{\frac{t-2}{2}+k+r} (1-R^2)^{\frac{n-t-2}{2}} d(R^2) = \frac{\Gamma\left(\frac{t}{2}+k+r\right)\Gamma\left(\frac{n-t}{2}\right)}{\Gamma\left(\frac{n}{2}+k+r\right)},$$

we obtain

$$\mu_{2k} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-1}{2}+k\right)} (1-\nu)^{\frac{n}{2}} \sum_{r=0}^{\infty} \frac{\Gamma^2\left(\frac{n}{2}+r\right)\Gamma\left(\frac{t}{2}+k+r\right)}{r!\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{t}{2}+r\right)\Gamma\left(\frac{n}{2}+k+r\right)} \nu^r \quad (9{\cdot}1).$$

In this we make the substitution

$$\frac{\Gamma\left(\frac{n}{2}+r\right)}{\Gamma\left(\frac{n}{2}+k+r\right)} = \frac{1}{\Gamma(k)} \int_0^1 x^{\frac{n}{2}+r-1} (1-x)^{k-1} dx \quad \ldots\ldots\ldots\ldots (9{\cdot}2);$$

upon reversing the order of integration and summation we have

$$\mu_{2k} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-1}{2}+k\right)\Gamma(k)}(1-\nu)^{\frac{n}{2}}$$

$$\times \int_0^1 (1-x)^{k-1} \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+r\right)\Gamma\left(\frac{t}{2}+k+r\right)}{r!\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{t}{2}+r\right)} x^{\frac{n}{2}+r-1} \nu^r dx.$$

Now make the further substitution

$$\frac{\Gamma\left(\frac{t}{2}+k+r\right)}{\Gamma\left(\frac{t}{2}+r\right)} = x^{-\frac{t}{2}-r+1} \frac{d^k}{dx^k} x^{\frac{t}{2}+k+r-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9{\cdot}3),$$

which gives, upon interchanging the order of summation and differentiation,

$$\mu_{2k} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-1}{2}+k\right)\Gamma(k)}(1-\nu)^{\frac{n}{2}}$$

$$\times \int_0^1 x^{\frac{n-t}{2}} (1-x)^{k-1} \frac{d^k}{dx^k}\left[x^{\frac{t}{2}+k-1} \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+r\right)}{r!\,\Gamma\left(\frac{n}{2}\right)} x^r \nu^r \right] dx.$$

The sum is now a binomial expansion and

$$\mu_{2k} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-1}{2}+k\right)\Gamma(k)}(1-\nu)^{\frac{n}{2}}$$

$$\times \int_0^1 x^{\frac{n-t}{2}} (1-x)^{k-1} \frac{d^k}{dx^k}\left[x^{\frac{t}{2}+k-1} (1-\nu x)^{-\frac{n}{2}} \right] dx \ldots\ldots (9{\cdot}4).$$

From this form, by k successive integrations by parts, it is easy in any particular case to calculate μ_{2k}. Thus for $k=1$ we have

$$\mu_2 = \frac{t-1}{n-1}(1-\nu)^{\frac{n}{2}} \int_0^1 x^{\frac{n-t}{2}} \frac{d}{dx}\left[x^{\frac{t}{2}} (1-\nu x)^{-\frac{n}{2}} \right] dx$$

$$= \frac{t-1}{n-1}(1-\nu)^{\frac{n}{2}} \left\{ (1-\nu)^{-\frac{n}{2}} - \frac{n-t}{2} \int_0^1 x^{\frac{n}{2}-1} (1-\nu x)^{-\frac{n}{2}} dx \right\}$$

$$= \frac{t-1}{n-1}\left\{ 1 - \frac{n-t}{n}(1-\nu)^{\frac{n}{2}} F\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}+1, \nu\right) \right\}.$$

Euler's transformation (10·1) of the hypergeometric reduces this to

$$\mu_2 = \frac{t-1}{n-1}\left\{1 - \frac{n-t}{n}(1-\nu)\,F\left(1, 1, \frac{n}{2}+1, \nu\right)\right\} \quad \text{.....(9·5),}$$

in which the series can easily be calculated to any required accuracy. For large samples the convergence is extremely rapid. A series of powers of n^{-1} may also be obtained by expanding each term of the hypergeometric series:

$$\mu_2 = \frac{t-1}{n-1}\left\{\nu + (1-\nu)\left[\frac{t-2\nu}{n} + \frac{2(t+2)\nu - 8\nu^2}{n^2} + \dots\right]\right\} \quad \text{.....(9·6).}$$

This form brings out the manner in which, for large samples, μ_2 varies with ν.

The moments may alternatively be found by a slightly different method, giving a general result in which an integral does not appear explicitly. In the identity (9·3) let x be replaced by ν, and make both this substitution and (9·2) in (9·1). The result is

$$\mu_{2k} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{n-1}{2}+k\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma(k)}\,\nu^{-\frac{t}{2}+1}(1-\nu)^{\frac{n}{2}} \times \frac{d^k}{d\nu^k}\int_0^1 x^{\frac{n}{2}-1}(1-x)^{k-1}\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+r\right)}{\Gamma\left(\frac{n}{2}\right)r!}x^r\nu^{\frac{t}{2}+k+r-1}\,dx.$$

The integral equals

$$\nu^{\frac{t}{2}+k-1}\int_0^1 x^{\frac{n}{2}-1}(1-x)^{k-1}(1-\nu x)^{-\frac{n}{2}}\,dx = \nu^{\frac{t}{2}+k-1}\left[\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(k)}{\Gamma\left(\frac{n}{2}+k\right)}\right]F\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}+k, \nu\right),$$

whence

$$\mu_{2k} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{n-1}{2}+k\right)\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{t-1}{2}\right)}\,\nu^{-\frac{t}{2}+1}(1-\nu)^{\frac{n}{2}} \times \frac{d^k}{d\nu^k}\left[\nu^{\frac{t}{2}+k-1}F\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}+k, \nu\right)\right] \quad \text{.....(9·7).}$$

This may be made even more explicit by performing the differentiation with the help of Leibnitz' theorem; and Euler's transformation may be applied to each of the hypergeometric functions to give a rapidly convergent series. In this way we obtain

$$\mu_{2k} = \sum_{r=0}^{k}\frac{k!}{r!\,(k-r)!}\,\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)\Gamma\left(\frac{t}{2}+k\right)\Gamma^2\left(\frac{n}{2}+k-r\right)}{\Gamma\left(\frac{n-1}{2}+k\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{t}{2}+k-r\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+2k-r\right)} \times \nu^{k-r}(1-\nu)^r\,F\left(k, k, \frac{n}{2}+2k-r, \nu\right) \quad \text{.....(9·8).}$$

The expressions obtained from this by substituting particular values of k are different in form from those obtained directly from (9·4), but are reducible to them with the help of the Gauss relations between "neighbouring" hypergeometric functions.

From (9·7) it is easy to see that $\mu_0 = 1$, as it should; this checks a long chain of deductions.

The asymptotic value of μ_{2k} for large values of n will now be investigated. In the expression for the kth derivative obtained from (9·4) by Leibnitz' theorem, the term of highest order in n is

$$x^{\frac{t}{2}+k-1} \nu^k \tfrac{n}{2}\left(\tfrac{n}{2}+1\right)\ldots\left(\tfrac{n}{2}+k-1\right)(1-\nu x)^{-\frac{n}{2}-k},$$

whence

$$\mu_{2k} \sim \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{n-1}{2}+k\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma(k)}(1-\nu)^{\frac{n}{2}}\nu^k \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)} \times \int_0^1 x^{\frac{n}{2}+k-1}(1-x)^{k-1}(1-\nu x)^{-\frac{n}{2}-k}\,dx$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)\Gamma^2\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n-1}{2}+k\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n}{2}+2k\right)\Gamma\left(\frac{n}{2}\right)}\nu^k(1-\nu)^{\frac{n}{2}}F\left(\tfrac{n}{2}+k, \tfrac{n}{2}+k, \tfrac{n}{2}+2k, \nu\right)$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+k\right)\Gamma^2\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n-1}{2}+k\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n}{2}+2k\right)\Gamma\left(\frac{n}{2}\right)}\nu^k F\left(k, k, \tfrac{n}{2}+2k, \nu\right)$$

$$\sim \frac{\Gamma\left(\frac{t-1}{2}+k\right)}{\Gamma\left(\frac{t-1}{2}\right)}\left(\frac{2\nu}{n}\right)^k \qquad \text{.....(9·9).}$$

Hence, for $s = t = 2$, the distribution of q approaches the normal form, with variance $\frac{\nu}{n}$, as is seen either from (9·9) or from (9·4), and mean value zero.

For $t \neq 2$, the distribution of q does not behave in this way. As in the case of multiple correlation, it is then confined to positive values. An approximation to the distribution is however suggested by the foregoing asymptotic values of the moments. These are in fact the moments of the χ^2 distribution with $t-1$ degrees of freedom, if we put

$$\chi^2 = \frac{nq^2}{\nu}.$$

The approximate distributions thus obtained may tentatively be used for testing the significance of q in large samples when ν has a value not too close to zero. For small samples and small values of ν, the methods of the next section are appropriate.

10. *The Distribution for Small Samples*. Form of the Frequency Curve.* The distribution of Section 8 may in certain cases be expressed in elementary forms. If $n-t$ is even, the Euler transformation

$$F(a, b, c, x) = (1-x)^{c-a-b} F(c-a, c-b, c, x) \quad \ldots\ldots\ldots\ldots (10{\cdot}1)$$

may be applied to the hypergeometric function in the distribution to give a terminating series, and the integration can then be carried out for each term, the integrand in (8·3) being a rational function of R, or involving (if t is odd) a single quadratic surd.

Consider for example the simple case $t=2$, $n=4$. Since $t=s$, it is more convenient to work with the distribution of q than with that of q^2. We halve the numerical coefficient because negative and positive sample values are distinguishable. The distribution (8·3) becomes, for positive values of q,

$$(1-\nu)^2\, dq \int_q^1 F(2, 2, 1, \nu R^2)\, dR.$$

Using (10·1), expanding $F(-1, -1, 1, R^2)$, and putting $\nu = \rho^2$, this becomes

$$(1-\rho^2)^2\, dq \int_q^1 (1-\rho^2 R^2)^{-3} (1+\rho^2 R^2)\, dR,$$

or, carrying out the integration,

$$\frac{(1-\rho^2)^2}{8\rho} \left\{ \log \frac{1-\rho q}{1-\rho} \frac{1+\rho}{1+\rho q} - \frac{2\rho q\,(3-\rho^2 q^2)}{(1-\rho^2 q^2)^2} + \frac{2\rho\,(3-\rho^2)}{(1-\rho^2)^2} \right\} dq.$$

Other special cases of the distribution function may be obtained by integrating the distributions obtained by R. A. Fisher for the multiple correlation coefficient for even values of n†.

A more systematic development, not depending on the oddness or evenness of n or t, and valuable when n and ν are not too great, is obtained by expanding the hypergeometric series in powers of ν and carrying out the integration term by term. Applying this procedure to (8·4) we obtain integrals of the form

$$\int_0^1 [x(1-x)]^{\frac{n-t-2}{2}} [1-(1-q^2)x]^{\frac{-n+t+1}{2}+k} dx \quad (k=0\ 1,\ 2, \ldots).$$

But this is itself a hypergeometric integral, and equals

$$\frac{\Gamma^2\left(\frac{m}{2}\right)}{(m-1)!} F\left(\frac{m-1}{2} - k,\ \frac{m}{2},\ m,\ 1-q^2\right),$$

where for brevity we put $m = n-t$ .....(10·2).

* The smallest samples to which the text is applicable are those for which $n=s+t$. For smaller samples, the matrix of pseudo-observations has more rows than columns; consequently there is a linear relation among the rows, i.e. among the sample variates, whose number is thus in effect reduced, so that a simpler theory is adequate. Thus, if $s=t=2$ and $n=3$, q reduces to a partial correlation coefficient, whose distribution is known.

† *Op. cit.*, p. 661.

Introducing the functions

$$v_k = \left(\frac{1+q}{2}\right)^{m-1} F\left(\frac{m-1}{2} - k,\ \frac{m}{2},\ m,\ 1-q^2\right) \quad \text{.....(10·3),}$$

the distribution (8·4) may thus be written, for $q > 0$,

$$\frac{(n-2)!}{(t-2)!\,(m-1)!}(1-\nu)^{\frac{n}{2}}(1-q)^{m-1}q^{t-2}dq \sum_{k=0}^{\infty} \frac{\Gamma^2\left(\frac{n}{2}+k\right)\Gamma\left(\frac{t}{2}\right)}{\Gamma^2\left(\frac{n}{2}\right)\Gamma\left(\frac{t}{2}+k\right)k!}\, v_k \nu^k$$

.....(10·4).

A factor $\frac{1}{2}$ must be applied to this expression in the case $t = 2$ if we then distinguish negative from positive values of q.

The first of the "relationes inter functiones contiguas" of Gauss* is

$$F(a-1, b, c, x) = \frac{[c - 2a + (a-b)x]F(a, b, c, x) + a(1-x)F(a+1, b, c, x)}{c-a}.$$

Putting $$a = \frac{m-1}{2} - k, \quad b = \frac{m}{2}, \quad c = m, \quad x = 1-q^2,$$

this shows that (10·3) satisfies the linear difference equation

$$v_{k+1} = \frac{(2k+1)(1+q^2)v_k + (m-2k-1)q^2 v_{k-1}}{m+2k+1} \quad \text{.....(10·5),}$$

which may be used as a recurrence relation for computing the successive v_k's as soon as we have determined two values whose indices differ by unity.

From the identity of Gauss (*ibid.*, p. 227)

$$F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}, x) = F\left(2\alpha, 2\beta, \alpha+\beta+\tfrac{1}{2}, \frac{1-\sqrt{1-x}}{2}\right)$$

we find at once

$$F\left(\frac{m-1}{2}, \frac{m}{2}, m, 1-q^2\right) = F\left(m-1, m, m, \frac{1-q}{2}\right) \quad \text{.....(10·6).}$$

In the series expansion of this last function, numerator and denominator factors cancel in each term in such a way as to leave a binomial expansion

$$\left(1 - \frac{1-q}{2}\right)^{-m+1} = \left(\frac{2}{1+q}\right)^{m-1}.$$

In this way we have, from (10·3) and (10·6),

$$v_0 = 1 \quad \text{.....(10·7).}$$

In (10·3) put $k = -1$, and apply (10·1). The result reduces, with the help of (10·6), to

$$v_{-1} = q^{-1}.$$

We have thus obtained two consecutive values of v_k, from which the rest are successively determined by means of (10·5), a relation which may also be written

$$v_{k+1} = q^2 v_{k-1} + \frac{2k+1}{m+2k+1}\{(1+q^2)v_k - 2q^2 v_{k-1}\} \quad \text{.....(10·8).}$$

* *Werke*, Vol. III. p. 130.

We thus find in turn

$$v_1 = q + \frac{(1-q)^2}{m+1}, \quad v_2 = q^2 + \frac{3}{m+3}(1-q)^2\left(q + \frac{1+q^2}{m+1}\right), \quad \ldots.$$

It is easy to show by means of the recurrence relation that the limit of v_k as m increases is q^k, and that the remaining terms of v_k constitute a polynomial in q having the factor $(1-q)^2$.

For a test of significance of q it is necessary to integrate the distribution from an arbitrary value to unity. For this purpose it is convenient to put $p = 1-q$. In terms of p,

$$v_1 = 1 - p + \frac{p^2}{m+1},$$

$$v_2 = 1 - 2p + \left(1 + \frac{3}{m+1}\right)p^2 - \frac{3}{m+1}p^3 + \frac{3}{(m+1)(m+3)}p^4,$$

$$v_3 = 1 - 3p + 3\frac{m+3}{m+1}p^2 - \frac{m+13}{m+1}p^3 + \frac{3(2m+11)}{(m+1)(m+3)}p^4 - \frac{15p^5}{(m+1)(m+3)} + \frac{15p^6}{(m+1)(m+3)(m+5)}.$$

The distribution (10·4) may, for the leading case $t = 2$, be written

$$(n-2)(1-\nu)^{\frac{n}{2}} p^{n-3}\left\{1 + \left(\frac{n}{2}\right)^2 v_1\nu + \left(\frac{n}{2}\right)^2\left(\frac{n+1}{2}\right)^2 \frac{v_2\nu^2}{(2!)^2} + \ldots\right\} dp \ldots (10{\cdot}9).$$

The series is uniformly convergent and may be integrated term by term, thus providing a test of significance for the tetrad difference. It is not however very convenient for computation unless n and ν are small. For large values of n, the method of the preceding section may be used: the standard error often gives a satisfactory test of significance, even when used with the crude inequality of Tchebycheff, which takes no account of the nature of the particular distribution.

Light is thrown on the form of the frequency curves by the expansions we have just obtained. The case $t = 2$ stands out as of a special character, different from the rest; this will be true in general where $s = t$. This special character is related to the fact that positive and negative sample values of q are distinguishable only if $s = t$. In other cases, just as in that of the multiple correlation (i.e. that of q when $s = 1$), the values must be taken as positive, and q^2 is in some respects a more natural variate to use.

The v_k, and therefore the convergent series in (10·4) and (10·9), and also the derivatives of the v_k and of the series, take definite finite values both for $q = 0$ and for $q = 1$. From (10·4) it is therefore evident that the frequency curve for q has, for $q = 1$, contact with the axis of order $m-2$. For $q = 0$ the ordinate of the curve is zero for $t \geqslant 3$, but has a finite value for $t = 2$.

The derivative with respect to q of the integral in (8·3) has, if $\nu < 1$, a finite negative value for $q = 0$ as well as for every positive value of q. The ordinate of the distribution curve for $t = 2$ will therefore have these properties. This curve must be symmetrical about $q = 0$. Hence it is not flat-topped, but has a corner above the origin.

But if $\nu = 1$ the distribution of q for $s = t = 2$ does not have such a discontinuity in the middle. For in this case linear functions of the variates in the two sets exist which are perfectly correlated with each other, and are thus for our purposes identical. Taking these as x_1 and x_3, (1·2) shows that q is in every sample the partial correlation of the remaining two variates. Hence when $\nu = 1$ the distribution becomes identically that of the partial correlation coefficient. According to R. A. Fisher's work*, this is the same as the distribution of the simple correlation coefficient, with the sample number reduced by unity, a distribution having continuous derivatives of all orders throughout its range.

11. *Tests for Complete Independence.* If $s = 2$ and *both* canonical correlations vanish, the normal distribution of the population implies complete independence between the two sets. No linear function of the first set is correlated with any of the second. In this case $\nu = 0$, and the distribution of q reduces, as is at once evident from the form (10·4), together with (10·2) and (10·7), to the extremely simple form,

$$\frac{(n-2)!}{(t-2)!\,(n-t-1)!}\, q^{t-2}(1-q)^{n-t-1}\,dq \quad \ldots\ldots\ldots\ldots\ldots\ldots (11{\cdot}1),$$

for positive values of q. Thus q has in this case the same distribution as the square of the multiple correlation coefficient in samples of $n\,(= N-1)$ from an uncorrelated normal population, with $t-1$ variates.

The question whether complete independence exists between two sets of variates for which we have sample correlations may be investigated by computing q and determining from (11·1) whether the probability of so great a value of q is negligible. This requires the integral of (11·1), which is easy to compute for any moderate value of t. For large values of t it may be obtained from the *Tables of the Incomplete Beta Function*†. For $t = 2$ the probability of a greater value of $|q|$ if complete independence really exists is simply

$$P = (1-|q|)^{n-2} = (1-|q|)^{N-3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11{\cdot}2),$$

where N is the number in the sample. In this way a very simple test for complete independence may be applied.

But this is not by any means the only possible test of complete independence between two sets. Indeed, the distribution of the vector alienation coefficient (Section 4),

$$z = \frac{D}{AB},$$

has been found by Wilks under this same hypothesis of complete independence and normality‡. This distribution, which was obtained by means of its moments, reduces for the case $s = 2$ which we are now studying to

$$\frac{(n-2)!}{2\,(t-1)!\,(n-t-2)!}\, z^{\frac{n-t-3}{2}}\,(1-\sqrt{z})^{t-1}\,dz \quad \ldots\ldots\ldots\ldots (11{\cdot}3).$$

* "The Distribution of the Partial Correlation Coefficient" in *Metron*, Vol. III. (1924), pp. 329—332.

† *Biometrika* Office, 1934.

‡ Wilks, *op. cit.*

The range of possible values of z is from 0 to 1, the latter corresponding to complete independence between the two sets of variates, just as does $q = 0$. q and z are not functionally related; for a continuum of values of either can be found which is consistent with any value of the other. The field of the joint distribution of the two is easily delimited by reference to the canonical correlations r_1 and r_2. Indeed, if we always take $q \geqslant 0$, we have from (4·10) that the field of variation of a point of coordinates q, z is in the quadrant in which both are positive, and is bounded by the parabola

$$z = (1-q)^2 \quad \text{.....(11·4),}$$

shown in Figure 1. The best agreement with the hypothesis of complete independence is shown by a sample for which $z = 1$ and $q = 0$, and which therefore corresponds to the point in the upper corner of the figure.

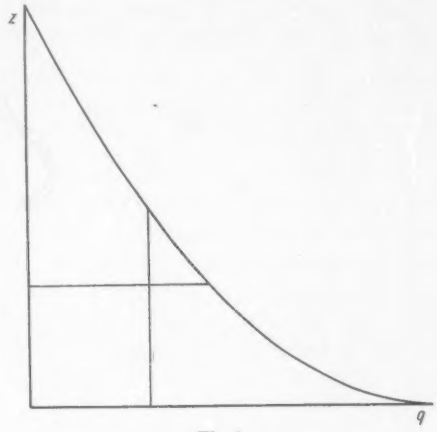

Fig. 1.

If we represent a sample by a point in a plane in which r_1 and r_2 are rectangular coordinates, and take r_1 as the greater, then the field of variation is the right triangle for which $0 \leqslant r_2 \leqslant r_1 \leqslant 1$. The point corresponding to best agreement with the hypothesis of complete independence is in this case the origin. The curves $q =$ constant and $z =$ constant, shown in Figure 2, are respectively hyperbolic arcs, and quartic curves which in the neighbourhood of the origin approximate circles. Their equations are

$$r_1 r_2 = q, \quad (1-r_1^2)(1-r_2^2) = z \quad \text{.....(11·5).}$$

To test complete independence by means of z, we need the integral of (11·3) from zero to the observed value. For $t = 2$ this is

$$P' = z^{\frac{n-3}{2}} \{(n-2)(1-\sqrt{z}) + \sqrt{z}\} \quad \text{.....(11·6).}$$

Like the integral of the distribution of q, that of (11·3) is easily found numerically from the Tables of the Incomplete Beta Function.

The existence of two different, though exact, tests of the same hypothesis makes us ask in what circumstances each should be used. No general answer appears to be possible to this question; but if we make sufficiently special assumptions about the nature of the deviations from our hypothesis that are likely to occur, or test for deviations of a sufficiently special character, a unique solution will exist.

In order that z differ from unity, it is seen from (11·5) to be sufficient for *either* of the canonical correlations to differ from zero. But in order that q differ from zero, it is necessary that *both* correlations differ from zero. This suggests that the z test will be the more sensitive to deviations from complete independence resulting from the existence of only a single component common to the two sets of variates; but if the correlations of one set with the other result from two independent common components operating to an approximately equal extent, the deviation from in-

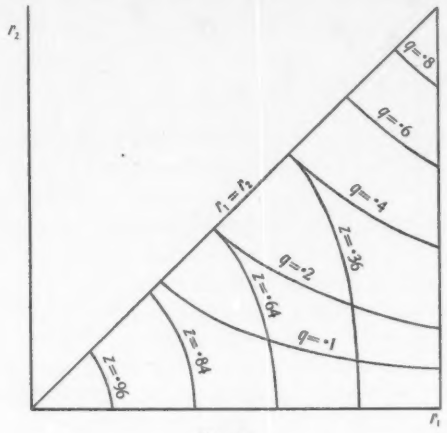

Fig. 2.

dependence will be revealed by q more clearly than by z. This conclusion is confirmed by a comparison of (11·2) with (11·6), putting for q and z their values from (11·5). If $r_2=r_1$, then

$$P=(1-r_1^2)^{n-2},\ P'=(1-r_1^2)^{n-2}\{(n-2)\,r_1^2+1-r_1^2\},$$

so that $P<P'$, and q provides the more sensitive test. If on the other hand $r_2=0$, $P=1$, so that q provides no evidence whatever of deviation from independence, though for a large enough sample P' becomes arbitrarily small, supplying evidence to any desired extent, if r_1 has any constant value other than zero.

Let us apply both tests for complete independence to Kelley's correlations cited in Section 6. For the correlations of arithmetical with reading abilities the values (6·1) of q and z were obtained, with $s=t=2$. From (11·2) the test for complete independence based on q gives $P=·023$, a probability so small that we may conclude that the two kinds of ability really have something in common.

The same conclusion is given even greater definiteness by the z test (11·6), from which we have $P' = \cdot 0001$.

The comparison of arithmetical with memory tests in Section 6 was for the values $s = 2$, $t = 3$. In this case we find from q that $P = 8{\cdot}6 \times 10^{-10}$, while the test for complete independence by means of z gives $P' = 1{\cdot}0 \times 10^{-24}$. Thus z gives a more sensitive test, and a more conclusive demonstration, of complete independence in both these cases than does q. The underlying reason for this is the considerable inequality between the two canonical correlations in each case.

One practical consideration in favour of the q test is that q is somewhat easier to calculate than z. The chief ground for distinction between them is however their sensitiveness to different types of deviations from complete independence.

Wilks in a later paper* derived the z test for complete independence from the likelihood criterion of Egon S. Pearson and J. Neyman. The considerations of this section therefore are relevant to an understanding of this criterion.

It is clear that a full understanding of the relations between two independent pairs of variates necessitates a knowledge of the bivariate distribution surfaces of r_1 and r_2, and of q and z. These we shall proceed to investigate.

12. *Alternants of a Plane and of a Sample.* The common method of specifying the orientation of a plane by the direction cosines of its normal is unsatisfactory in a space of more than three dimensions, since then a plane has an infinity of normals at each point. Instead we shall use determinants which may be regarded as of the form known as alternants. If in a space of n dimensions a flat space of k dimensions is determined by the origin and k other points, we shall call the k-rowed determinants in the matrix of the rectangular coordinates of these k points the *alternants* of the k-space. The alternants are C_k^n in number, and are connected by numerous quadratic relations. The number of degrees of freedom of the k-space through a fixed point is $k(n-k)$. The alternants depend only on the k-space, and not on the particular points used to determine it, except for multiplication of all the alternants by a constant; for to replace the k points by k others in the same k-space is to replace their matrix of coordinates by a new one whose rows are linear functions of the old; and this merely multiplies all the k-rowed determinants by a constant.

Taking the case $k = 2$, a plane containing the points

$$x_1, x_2, \ldots, x_n,$$

$$y_1, y_2, \ldots, y_n$$

is specified by the alternants

$$p_{ij} = x_i y_j - x_j y_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12{\cdot}1),$$

which are analogous to the Plücker coordinates of a line. Indeed, the planes through a point in n-space are in one-to-one correspondence with the lines in which they meet an $(n-1)$-space not containing the point.

* "On the Independence of k Sets of Normally Distributed Statistical Variables" in *Econometrica*, Vol. III. (1935), pp. 309—326.

The relations connecting these alternants, apart from the obvious relations

$$p_{ij} = -p_{ji} \quad \text{.....(12·2),}$$

are obtainable from the fact that, on account of identical rows,

$$\begin{vmatrix} x_i & x_j & x_k & x_m \\ y_i & y_j & y_k & y_m \\ x_i & x_j & x_k & x_m \\ y_i & y_j & y_k & y_m \end{vmatrix} = 0.$$

Applying a Laplace expansion to the first two rows of this determinant we obtain

$$p_{ij}p_{km} - p_{ik}p_{jm} + p_{im}p_{jk} = 0 \quad \text{.....(12·3).}$$

There is one of these relations for each combination of subscripts, but not all are independent. To obtain a set of independent relations which shall imply all the rest, we first observe that not all the p_{ij} can be zero if a definite plane is to be specified, for in that case the x- and y-points would be collinear with the origin. Let the notation be arranged so that $p_{12} \neq 0$. Then in terms of

$$\left.\begin{matrix} p_{12}, p_{13}, p_{14}, \ldots, p_{1n} \\ p_{23}, p_{24}, \ldots, p_{2n} \end{matrix}\right\} \quad \text{.....(12·4)}$$

any other alternant p_{ij} is determined from (12·3), by putting $k = 1$, $m = 2$, so that

$$p_{ij} = \frac{p_{1i}p_{2j} - p_{1j}p_{2i}}{p_{12}} \quad \text{.....(12·5).}$$

The relations (12·5) constitute a complete set of independent relations of the form (12·3). For if in the left member of (12·3) we substitute for all the alternants the expressions obtained from (12·5) by putting the several combinations of subscripts in place of i and j, the resulting equation is satisfied identically. Furthermore, any set of quantities p_{ij} $(i, j = 1, \ldots, n)$ satisfying (12·2) and (12·5), and not all zero, determines uniquely a plane through the origin. For, supposing that $p_{12} \neq 0$, we may obtain the coordinates of two points not collinear with the origin in the following manner. Let $x_1 = y_2 = 0$, and let $x_2 = 1$. Putting $i = 1$, $j = 2$ in (12·1) we then have $y_1 = -p_{12}$. Then putting first $j = 1$ and then $j = 2$, we obtain $x_i = \frac{p_{i1}}{y_1} = \frac{-p_{i1}}{p_{12}}$; and $y_i = p_{2i}$. The points whose coordinates are thus determined cannot be collinear with the origin, for if they were the alternants would all be zero.

Since the alternants of a plane have so far been determined only to within a multiplicative constant, we may determine them uniquely if we add the condition

$$\Sigma p_{ij}^2 = 1 \quad \text{.....(12·6).}$$

Here we use the sign Σ to mean summation over the $\frac{n(n-1)}{2}$ alternants

$$p_{12}, p_{13}, \ldots, p_{n-1,n}$$

for which the first subscript is less than the second. This condition on the quantities (12·4) shows that the number of independent alternants is $2n - 4$, which is the number of degrees of freedom of the plane.

We define the alternants of a set of observations of two variates on n individuals (or rather with n degrees of freedom after elimination of the mean and possibly other variables and an appropriate orthogonal transformation of the observations) as the determinants p_{ij} in the matrix of observations, or of pseudo-observations, multiplied by a constant. It will frequently be convenient to choose this constant so that (12·6) is satisfied. This definition breaks down in the case where the determinants are all zero, but if the observations are a sample from a continuous distribution this is infinitely improbable, and we shall disregard this case.

It is clear that all relations between two pairs of variates that are invariant under internal linear transformations of the pairs, and are based on a sample, must be expressible in terms of the alternants of the two pairs; for such relations must correspond to relations between planes through the origin in n-space, independent of the particular points used to define the planes. The relations depending on correlations must also be invariant under rotations of the n-space about the origin, since the correlations are cosines of angles at the origin, which are invariant under rotations. We shall therefore suppose for simplicity that the axes have been rotated in such a way that the first two of them lie in the plane of the observations on one pair of variates, and contain the observation points for this pair. We shall suppose further that all the observation points are at unit distance from the origin, so that the sum of the squares of the observations on each variate is unity. None of these assumptions reduces the generality of our results. The matrix of observations now takes the form

$$\begin{Vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & \dots & x_n \\ y_1 & y_2 & y_3 & y_4 & y_5 & \dots & y_n \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \end{Vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (12\cdot7).$$

The determinant D of the correlations among the four variates is the determinant of sums of squares and products, since each sum of squares is unity. Hence, by Section 2, D is the sum of the squares of the four-rowed determinants in the matrix (12·7). But all these determinants are zero except those containing the first and second columns. The determinant consisting of the first, second, ith and jth columns equals $x_i y_j - x_j y_i$. Defining this as p_{ij}, we obtain the result

$$D = \Sigma' p_{ij}^2 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (12\cdot8),$$

where Σ' denotes summation from 3 to n with respect to i, and from $i+1$ to n with respect to j. It is further evident from (12·7) that the determinants of correlations within the sets are

$$A = \begin{vmatrix} 1 & r_{12} \\ r_{12} & 1 \end{vmatrix} = \Sigma p_{ij}^2, \; B = 1 \quad \dots\dots\dots\dots\dots\dots\dots\dots (12\cdot9).$$

The equation (3·6) for the canonical correlations of the sample is

$$\begin{vmatrix} \lambda & \lambda r_{12} & r_{13} & r_{14} \\ \lambda r_{12} & \lambda & r_{23} & r_{24} \\ r_{13} & r_{23} & \lambda & \lambda r_{34} \\ r_{14} & r_{24} & \lambda r_{34} & \lambda \end{vmatrix} = 0 \quad \dots\dots\dots\dots\dots\dots (12\cdot10).$$

The coefficient of λ^4 in this equation is $AB = \Sigma p_{ij}^2$. The term independent of λ is the square of the sum of the products of the determinants in the first two rows of (12·7) by the corresponding determinants in the last two rows. The latter determinants, however, are all zero but the first; hence the constant term in the equation is p_{12}^2. The coefficient of λ^2 may be obtained by putting $\lambda = 1$ in the left member of (12·10) and subtracting the coefficient of λ^4 and the constant term. This coefficient is therefore equal to

$$D - AB - p_{12}^2 = -\Sigma p_{1i}^2 - \Sigma p_{2i},$$

where Σ denotes summation with respect to i from 1 to n. We recall in this connection that (12·2) shows that $p_{ii} = 0$. The equation may thus be written in terms of alternants

$$\lambda^4 \Sigma p_{ij}^2 - \lambda^2 \Sigma (p_{1i}^2 + p_{2i}^2) + p_{12}^2 = 0 \quad \text{.....(12·11).}$$

If we regard this as a quadratic equation in λ^2, the roots are r_1^2 and r_2^2. Hence, from (11·5) and the expressions for the coefficients in terms of the roots,

$$q^2 = \frac{p_{12}^2}{\Sigma p_{ij}^2}, \quad z = 1 - \frac{\Sigma (p_{1i}^2 + p_{2i}^2) - p_{12}^2}{\Sigma p_{ij}^2} = \frac{\Sigma' p_{ij}^2}{\Sigma p_{ij}^2} \quad \text{.....(12·12).}$$

If we adopt the further convention (12·6), which by (12·9) is seen to be equivalent to assuming that the first pair of variates has been reduced by an internal transformation so that the correlation is zero, (12·11) and (12·12) simplify to

$$\lambda^4 - \lambda^2 \Sigma (p_{1i}^2 + p_{2i}^2) + p_{12}^2 = 0 \quad \text{.....(12·13),}$$

$$q = \pm p_{12}, \quad z = \Sigma' p_{ij}^2 \quad \text{.....(12·14).}$$

If we do not specialise one of our planes with reference to the coordinates, but take its alternants as q_{ij}, while those of the other are p_{ij}, it is easy to see in the foregoing manner, or from Section 4, that the vector correlation is

$$q = \frac{\Sigma p_{ij} q_{ij}}{\sqrt{(\Sigma p_{ij}^2)(\Sigma q_{ij}^2)}} \quad \text{.....(12·15).}$$

This has the form of the ordinary formula for a correlation coefficient, or of the cosine of an angle. From the latter fact comes the following conclusion, which is of the utmost importance for our purposes.

Let us take the alternants p_{ij} for which $i < j$ as Cartesian coordinates in a space of $\frac{n(n-1)}{2}$ dimensions, and denote by V the subspace in which the equations (12·5) and (12·6) hold. Then V is a curved space of $2n - 4$ dimensions, in which all the equations (12·3) hold, since they follow from (12·5). The points of V are in one-to-one correspondence with the planes through the origin in n-space. A property of this correspondence which we shall use is that it is *metrical*, in the sense that any rotation of the n-space about the origin engenders a transformation of V which is also a rotation. This fact follows from (12·15), which shows that q is the cosine of the angle at the origin between lines extending to points of V representing two planes. Under a rigid rotation of the n-space, the correlations defining q are all

invariant, so that q is invariant. Hence the points of V representing the rotated pair of planes are exactly as far apart as the points representing the planes in their original position, since all these points are, by (12·6), equidistant from the origin. Thus the transformation of V satisfies the definition of a rotation.

13. *The Bivariate Distribution for Complete Independence* ($s=t=2$, $n=4$). If there is complete independence of one pair of variates from another, $\rho_1=\rho_2=0$. We may without loss of generality regard the internal correlations also as zero. The planes corresponding to a sample from a normal distribution are then determined by lines drawn through the origin in n-space at random, in the sense that the probability of a line meeting any region on a surrounding sphere is proportional to the generalized area of the region. The chance selection of a plane in this way is equivalent to the selection of a point in V in such a way that the element of probability is proportional to the volume element. For, since any plane through the origin in n-space can be rotated into any other, any point in V can be rotated into any other, and will carry with it in this rotation its probability density, which must therefore be uniform over the whole of V. Thus all problems of finding distributions of statistics calculated from the pairs of variates in such a way as to be invariant under internal transformations reduce to purely geometrical problems of finding the $(2n-4)$-dimensional volumes of the corresponding regions in V.

The distribution of q and z, or of r_1 and r_2, will be deduced with the help of methods of parametric representation resembling those previously applied by the author to other statistical problems*. First we take the case $n=4$. In the six-dimensional space in which the alternants are Cartesian coordinates, V is then a curved four-dimensional space having the equations

$$p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23}=0, \quad p_{12}^2+p_{13}^2+p_{14}^2+p_{23}^2+p_{24}^2+p_{34}^2=1 \qquad \text{.....(13·1).}$$

It follows that V may be defined in terms of four parameters α, β, γ, δ by means of the equations

$$\left.\begin{aligned} p_{14} &= \tfrac{1}{2}(\sin\alpha\sin\beta+\sin\gamma\sin\delta)\\ p_{23} &= \tfrac{1}{2}(\sin\alpha\sin\beta-\sin\gamma\sin\delta)\\ p_{13} &= \tfrac{1}{2}(\cos\alpha\sin\beta+\cos\gamma\sin\delta)\\ p_{24} &= -\tfrac{1}{2}(\cos\alpha\sin\beta-\cos\gamma\sin\delta)\\ p_{12} &= \tfrac{1}{2}(\quad\cos\beta+\quad\cos\delta)\\ p_{34} &= \tfrac{1}{2}(\quad\cos\beta-\quad\cos\delta) \end{aligned}\right\} \qquad \text{.....(13·2),}$$

since these equations satisfy both the equations (13·1). All points of V are included when we allow α and γ to vary from 0 to 2π, and β and δ from 0 to π. The element of volume in V is of course

$$\sqrt{g}\,d\alpha\,d\beta\,d\gamma\,d\delta,$$

* "The Distribution of Correlation Ratios Calculated from Random Data" in *Proceedings of the National Academy of Sciences*, Vol. XI. (1925), pp. 657—662; "The Generalization of Student's Ratio" in *Annals of Mathematical Statistics*, Vol. II. (1931), pp. 360—378; "The Physical State of Protoplasm," *loc. cit.*

where g is the sum of the squares of the four-rowed determinants in the matrix of partial derivatives of the p_{ij} with respect to α, β, γ and δ. These derivatives are the halves of the elements of

$$\begin{Vmatrix} \cos\alpha\sin\beta & \cos\alpha\sin\beta & -\sin\alpha\sin\beta & \sin\alpha\sin\beta & 0 & 0 \\ \sin\alpha\cos\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta & -\cos\alpha\cos\beta & -\sin\beta & -\sin\beta \\ \cos\gamma\sin\delta & -\cos\gamma\sin\delta & -\sin\gamma\sin\delta & -\sin\gamma\sin\delta & 0 & 0 \\ \sin\gamma\cos\delta & -\sin\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta & -\sin\delta & +\sin\delta \end{Vmatrix}.$$

The sum of products of corresponding elements in each pair of rows of this matrix is zero; the sums of squares are respectively $2\sin^2\beta$, 2, $2\sin^2\delta$, and 2, each of which sums must be divided by 4. Thus in accordance with Section 2 we have

$$g = \tfrac{1}{16}\sin^2\beta\sin^2\delta,$$

so that the volume element in V is

$$\tfrac{1}{4}\sin\beta\sin\delta\,d\alpha\,d\beta\,d\gamma\,d\delta, \quad \text{or} \quad \tfrac{1}{4}d(\cos\beta)\,d(\cos\delta)\,d\alpha\,d\gamma.$$

From this it follows, if we put

$$\xi = \cos\beta, \quad \xi' = \cos\delta \qquad \text{.....(13·3),}$$

that ξ and ξ' are independently distributed with uniform density between -1 and 1. If we take q to have the sign of p_{12}, and $\sqrt{z}$ to have that of p_{34}, we have, from (12·14), (13·2) and (13·3),

$$q = p_{12} = \tfrac{1}{2}(\xi + \xi'), \quad \sqrt{z} = p_{34} = \tfrac{1}{2}(\xi - \xi'),$$

whence

$$\xi = q + \sqrt{z}, \qquad \xi' = q - \sqrt{z}.$$

These relations show, because they are linear, and since points of coordinates (ξ, ξ') are uniformly distributed in the square $\xi = \pm 1$, $\xi' = \pm 1$, that points of coordinates $(q, \sqrt{z})$ are uniformly distributed in the square bounded by the four lines

$$q \pm \sqrt{z} = \pm 1.$$

If we restrict q and $\sqrt{z}$ to positive values, their distribution will be

$$2\,dq\,d\sqrt{z},$$

within the triangle bounded by the coordinate axes and the line $q + \sqrt{z} = 1$. The distribution of q and z is therefore

$$z^{-\frac{1}{2}}dz\,dq \qquad \text{.....(13·4),}$$

subject to the limitations that both are positive, and that

$$z \leqslant (1-q)^2 \qquad \text{.....(13·5).}$$

If we integrate (13·4) with respect to q or z we obtain Wilks' distribution (11·3) of z, or the distribution (11·1) of q, respectively, for the case $s = t = 2$, $n = 4$.

We may regard (α, β) and (γ, δ) as the spherical coordinates of two points on a sphere in 3-space. The equations (13·2) thus establish a correspondence having metrical properties between planes through a point in 4-space and pairs of points

on an ordinary sphere. In this representation q appears as the mean distance of the two points from a fixed plane, while $\sqrt{z}$ is half the difference of the distances from this plane. From the theorem of Archimedes that the area of a zone depends only on the distance between the bounding planes and the radius of the sphere (which in this case is unity), it is therefore evident that when z is fixed the distribution of q is of uniform density, confirming (13·4).

Let us call r the cosine of the angle between the two lines determining our variable plane in 4-space. The distribution of r, which is the sample correlation between two really uncorrelated variates, is readily seen geometrically, or by putting $n=4$ in the general distribution of such sample correlations, to be

$$\frac{2}{\pi}(1-r^2)^{\frac{1}{2}}\,dr.$$

Moreover, this distribution is independent of that of q and z, since r depends only on the angle within the plane, and q and z on the plane itself. Consequently the joint distribution of the three is, for $n=4$,

$$\frac{2}{\pi}(1-r^2)^{\frac{1}{2}}z^{-\frac{1}{2}}\,dr\,dz\,dq \qquad (13{\cdot}6).$$

This result and the following theorem will be used in Section 15 in extending the distribution to a general value of n.

14. *Theorem on Circularly Distributed Variates.* The sum or difference of two variates distributed independently and with uniform density over a particular range is known to have a distribution represented by an isosceles triangle whose base has double the breadth of the original range. If however each value of the sum or difference is reduced with the original range as modulus—that is, is replaced by the remainder after dividing by the range—the resulting distribution is exactly the original one, with uniform density over the same range. This is a special case of the following rather remarkable

THEOREM: *If any number of variates are distributed independently and with uniform density from* 0 *to a, then any linear function of these variates with integral coefficients, when reduced modulo a, is likewise distributed with uniform density from* 0 *to a. Any number of such functions, if algebraically independent, are also independent in the probability sense.*

The truth of this theorem becomes evident when we regard each set of values of the variates as a point in a space having the metrical properties of a hypercube of as many dimensions as there are variates, but with a topological nature determined by making each pair of diametrically opposite faces of the hypercube correspond to a single region of the space. The space is thus a closed manifold generalizing a torus in its topology, but not contained in a euclidean space, because of its metrical nature. For two variates this representing space would be approximated by a torus obtained by revolving a very small circle about a very distant line in its plane. Another representation in this case would be by means of the

squares of side a into which a plane is divided by two sets of parallel lines, all points occupying a particular position within their respective squares being regarded as identical. If we call the variates, or coordinates, $x_1, x_2, \ldots, x_m$, the linear functions

$$y_i = \sum_{j=1}^{m} a_{ij} x_j \ (i = 1, 2, \ldots, k) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14{\cdot}1),$$

in which the coefficients a_{ij} are positive or negative integers or zero (but are not all zero for any value of i), are constants over loci which, on the representation on a plane or flat space of m dimensions, are parallel lines, or hyperplanes of $m-1$ dimensions. In the space itself, in which only one point corresponds to each set of values of the variates other than 0 and a, the loci (14·1) are closed curves, or closed hypersurfaces, because the coefficients are integers. It is obvious that the volume in this space contained between $y_i = b$ and $y_i = c$ must, on account of the homogeneity, be proportional to $b - c$.

If the k linear functions (14·1) are linearly independent, the loci obtained by giving each y_i a succession of constant values differing consecutively by $\frac{a}{p}$, where p is an integer, will divide the space into congruent parallelepipedons. If $k-1$ of the y_i are constrained to lie in certain of these intervals, the representing point is merely constrained to lie in a certain layer. Since all such layers must be congruent, the distribution of the kth of the y_i, reduced modulo a, is not affected by this constraint. Hence all the variates thus reduced are independent.

15. *Generalization of Section* 13 *for Samples of Any Size.* No direct extension to a larger number of dimensions of the method of using alternants in Section 13 appears to lead in any simple fashion to the generalization of the distribution there found. This generalization will however be obtained with the help of hyperspherical coordinates in the space of the observations.

On account of the spherical symmetry of the density distributions in n-space in the absence of true correlation, our distributions will not be affected if we assume the two points of coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ to be taken independently at random on a unit sphere about the origin in n dimensions, in such a way that the element of probability for each is proportional to the element of $(n-1)$-dimensional area on this hypersphere. If we define the hypersphere parametrically by the equations

$$\left.\begin{aligned}
x_1 &= \sin\theta_1 \sin\theta_2 \\
x_2 &= \cos\theta_1 \sin\theta_2 \\
x_3 &= \cos\theta_2 \cos\theta_3 \\
x_4 &= \cos\theta_2 \sin\theta_3 \cos\theta_4 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{n-1} &= \cos\theta_3 \sin\theta_3 \ldots \sin\theta_{n-2} \cos\theta_{n-1} \\
x_n &= \cos\theta_2 \sin\theta_3 \ldots \sin\theta_{n-2} \sin\theta_{n-1}
\end{aligned}\right\} \quad \ldots\ldots (15{\cdot}1),$$

$$(0 \leqslant \theta_1, \theta_{n-1} \leqslant 2\pi;\ 0 \leqslant \theta_2 \leqslant \pi/2;\ 0 \leqslant \theta_3, \theta_4, \ldots, \theta_{n-2} \leqslant \pi);$$

which satisfy $\Sigma x^2 = 1$ identically, then the element of $(n-1)$-dimensional area for the x-point may be written

$$\sqrt{g}\, d\theta_1 \theta_2 \dots d\theta_{n-1},$$

where g is a determinant of $n-1$ rows, in which the element in the ith row and jth column is

$$\sum_{a=1}^{n} \frac{\partial x_a}{\partial \theta_i} \frac{\partial x_a}{\partial \theta_j} \quad \dots\dots\dots\

it is evident from (15·1), (15·4), (15·6) and (15·8) that

$$r = \Sigma xy = \cos(\theta_1 - \phi_1)\sin\theta_2 \sin\phi_2 + \cos\lambda\cos\theta_2\cos\phi_2 \ldots\ldots\ldots(15{\cdot}9).$$

The sample values of the variates of the second pair may in the absence of correlation in the population be represented by an arbitrary pair of fixed lines; we shall represent them by the first two coordinate axes. We then have from (12·12) and (12·1), with the help of Section 2,

$$q^2 = \frac{(x_1y_2 - x_2y_1)^2}{\Sigma x^2 \Sigma y^2 - (\Sigma xy)^2}, \quad z = \frac{\Sigma' x^2 \Sigma' y^2 - (\Sigma' xy)^2}{\Sigma x^2 \Sigma y^2 - (\Sigma xy)^2},$$

where Σ denotes summation from 1 to n, and Σ' from 3 to n. These expressions become, upon substitution from (15·1), (15·4), (15·6), (15·8) and (15·9),

$$q^2 = \frac{\sin^2(\theta_1 - \phi_1)\sin^2\theta_2 \sin^2\phi_2}{1 - r^2}, \quad z = \frac{\cos^2\theta_2 \cos^2\phi_2 \sin^2\lambda}{1 - r^2} \ldots\ldots(15{\cdot}10).$$

To simplify the notation we shall replace θ_2 and ϕ_2 simply by θ and ϕ respectively. We shall also put

$$\omega = \theta_1 - \phi_1.$$

Since only the sines and cosines of ω will enter into our discussion, we may regard ω as reduced modulo 2π. Now θ_1 and ϕ_1 vary independently and, as is seen from (15·3) and (15·5), with uniform density from 0 to 2π. Hence their difference ω must, by the theorem of the last section, have a distribution of uniform density from 0 to 2π. Moreover, since θ_1 and ϕ_1 have been seen to be independent of θ, ϕ and λ, it follows that ω is likewise independent of them; indeed, ω, θ, ϕ and λ constitute a completely independent set. The distributions of θ and ϕ are determined by integrating (15·3) and (15·5) between constant limits with respect to all the variates appearing in them except θ_2 and ϕ_2 respectively. Combining with (15·7) the result of this integration and the uniformity of the distribution of ω, we have that the element of probability is of the form

$$k_n \sin\theta \cos^{n-3}\theta \sin\phi \cos^{n-3}\phi \sin^{n-4}\lambda\, d\theta\, d\phi\, d\lambda\, d\omega \quad \ldots\ldots\ldots(15{\cdot}11),$$

where k_n depends only on n. The limits for θ and ϕ are 0 and $\frac{\pi}{2}$; for λ they are 0 and π; for ω they are 0 and 2π.

In the new notation, (15·9) and (15·10) become

$$r = \cos\omega \sin\theta \sin\phi + \cos\lambda \cos\theta \cos\phi \quad \ldots\ldots\ldots\ldots(15{\cdot}12),$$

$$q^2 = \frac{\sin^2\omega \sin^2\theta \sin^2\phi}{1 - r^2}, \quad z = \frac{\sin^2\lambda \cos^2\theta \cos^2\phi}{1 - r^2} \ldots\ldots\ldots(15{\cdot}13).$$

We next consider a transformation to the variates q, z, r and ω. Without troubling to compute the Jacobian J of this transformation, we need observe only that it is independent of n, since the functional relations (15·12) and (15·13) do not involve n. Substituting in (15·11) from the second of the equations (15·13) we find that the distribution is of the form

$$k_n \psi z^{\frac{n-3}{2}} (1 - r^2)^{\frac{n-3}{2}}\, dq\, dz\, dr\, d\omega,$$

where ψ does not involve n. Upon integrating this with respect to ω between certain limits depending on q, z and r, but not on n, we have the distribution

$$k_n \Psi z^{\frac{n-3}{2}} (1-r^2)^{\frac{n-3}{2}} dq\,dz\,dr,$$

which for $n=4$ must reduce to (13·6). Comparing with (13·6) we have $k_4 \Psi = \frac{2}{\pi z}$. Inasmuch as the distribution of r is known to be

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}} (1-r^2)^{\frac{n-3}{2}} dr,$$

and to be independent of that of q and z, we have for the distribution of the latter two

$$h_n z^{\frac{n-5}{2}} dz\,dq,$$

where h_n depends only on n. Since the integral over the entire range of variation defined by the inequalities

$$0 \leqslant q \leqslant 1, \quad 0 \leqslant z \leqslant 1, \quad z \leqslant (1-q)^2$$

must be unity, the constant h_n is readily found. The distribution is

$$\tfrac{1}{2}(n-2)(n-3) z^{\frac{n-5}{2}} dz\,dq \quad \text{.....(15·14).}$$

Its form shows that, in the plane of Fig. 1, the loci of uniform density are horizontal lines.

The distribution of the canonical correlations is determined by (15·14), together with (11·5), which latter gives

$$\frac{\partial(q, z)}{\partial(r_1, r_2)} = 2(r_1^2 - r_2^2).$$

Thus the distribution of the correlations in case of complete independence is

$$(n-2)(n-3)(r_1^2 - r_2^2)(1-r_1^2)^{\frac{n-5}{2}} (1-r_2^2)^{\frac{n-5}{2}} dr_1\,dr_2 \text{.....(15·15).}$$

16. *Further Problems.* The foregoing treatment of sampling distributions is obviously incomplete. It would be desirable to have exact distributions, both of sample canonical correlations and of various functions of them, for cases in which the canonical correlations in the population have arbitrary values. The coefficients obtained for the canonical variates have sampling distributions which remain to be determined. Furthermore, various possible comparisons among different samples remain to be investigated; for example, there is the problem of testing the significance of the difference between vector correlations obtained from different samples.

A generalization of the problem of relations between two sets of variates invariant under internal linear transformations is that of invariants under such transformations of three or more sets of variates. A beginning of this theory has been made by

Wilks in the work previously alluded to; the z we have used is only a special case of a statistic of his, which in general is defined with reference to any number of sets of variates as a fraction, whose numerator is the determinant of the correlations among all the variates, and whose denominator is the product of the determinants of correlations within sets. It is obvious also that the invariants we have discussed, taken between every two of the sets, are invariants of such a system. An additional set of invariants will be the roots of the equation in λ resembling (3·6), obtained from the determinant of all the correlations or covariances by multiplying those between variates in the same set by $-\lambda$. It is easy to prove with the help of the theory of λ-matrices that the roots and coefficients of this equation are actually invariants.

A generalization of our work in a different direction would consider invariants, not under all linear internal transformations, but under a restricted class of these transformations. For example, a study of the relations of the prices to the quantities of several commodities might well consider transformations of *commodities*, such for example as the mixing of different grades of wheat, or the combination of raw materials and labour into finished products. If from quantities $q_1, q_2, \ldots$ of the old commodities there are formed quantities $q_1', q_2', \ldots$ of the new, we may, at least approximately, write

$$q_i' = \sum_j c_{ij} q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16\cdot1).$$

If all the costs and profits of the mixing or manufacturing operation are regarded as prices of constituents, the *value* of one set of commodities will equal that of the other, so that

$$\Sigma p_i' q_i' = \Sigma p_j q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16\cdot2),$$

where the p_j are the prices of the original commodities and the p_i' are those of the products. If we regard (16·1) as a linear transformation of the quantities, there will be a corresponding linear transformation of the prices, whose coefficients may be determined in terms of the c_{ij} by substituting (16·1) and

$$p_i' = \Sigma d_{ik} p_k$$

in (16·2) and then equating coefficients of like terms. This process shows that

$$\Sigma c_{ij} d_{ik} = \delta_{jk}, \quad = 1 \text{ if } j = k, \quad = 0 \text{ if } j \neq k.$$

These equations fully determine the d_{ik} as functions of the c_{ij}. The relation is such that the transformation of prices is *contragredient* to that of quantities.

An important class of relations between prices and quantities of a group of commodities would be the class of relations invariant under mixings of the kind described above. The canonical correlations and their functions, which are the main subject of this paper, are such invariants. But on account of the restriction that linear transformations of one set of variates shall be contragredient to those of the other, there will be additional invariants for this case, which remain to be investigated.

Other important problems are connected with the case of equal canonical correlations, which had to be excluded in deriving the approximate standard errors in Section 5. If two or more canonical correlations in the population are equal, it appears that the distribution of the corresponding sample values does not approach the multivariate normal form. This case is of much practical importance, owing to the practice of devising tests designed to measure the same character with equal accuracy. The psychologists' use of "reliability coefficients," and of "correlations corrected for attenuation" has been recognized as unsatisfactory. One symptom of trouble is that the formula for correlations corrected for attenuation sometimes gives values greater than unity. A satisfactory treatment of this difficulty should be possible with the help of the distribution function, when found, of sample canonical correlations, or of the vector correlation, when in the population the roots of the determinantal equation are equal.

A PRELIMINARY STUDY OF THE ERUPTION OF THE MANDIBULAR THIRD MOLAR TOOTH IN MAN BASED ON MEASUREMENTS OBTAINED FROM RADIOGRAPHS, WITH SPECIAL REFERENCE TO THE PROBLEM OF PREDICTING CASES OF ULTIMATE IMPACTION OF THE TOOTH.

By C. BOWDLER HENRY, M.R.C.S., L.D.S. and G. M. MORANT, D.Sc.

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(1) *Introduction: Clinical and Anatomical Aspects of Mal-eruption of the Mandibular Third Molar.* One of the major problems in clinical stomatology is the menace of the mal-erupted mandibular third molar, commonly known as the impacted wisdom tooth. The lesions to which this abnormality gives rise may be local, or referred, or general, and they vary in intensity; but they may, and often do, lead to incapacitating pain or severe illness, and sometimes even to death.

An analysis of cases of abnormality and complications of wisdom teeth, upper and lower together, was made from 622 collected reports*. Of this total, 311 were communicated privately by practitioners in medicine, surgery and dentistry, from their case notes, and they were set side by side for comparison with an equal number of cases extracted, without selection, from the literature. The accompanying chart (Fig. 1) shows the frequencies for the more important conditions noted in each of the two series. The numerals in brackets give the numbers of cases in which the symptom or condition was cardinal. It must be admitted, however, that

* C. Bowdler Henry: "Wisdom Teeth and their Complications." Hunterian Lecture (abridged) delivered at the Royal College of Surgeons of England. *The Lancet*, Feb. 9, 1935, p. 313.

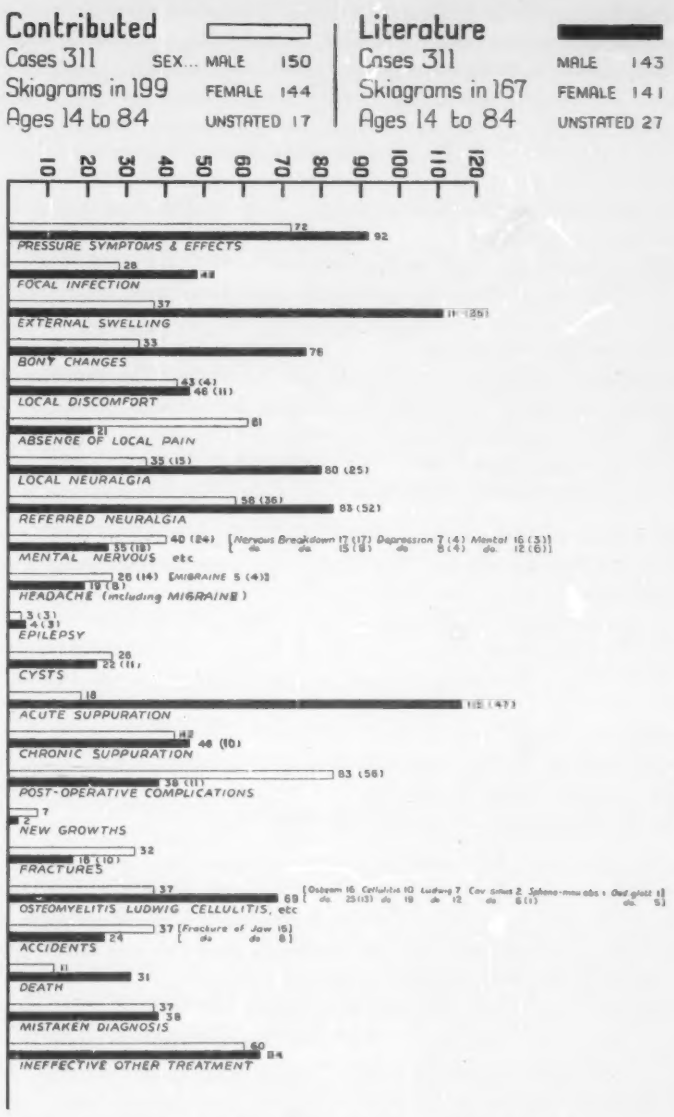

Fig. 1. *Chart of Complications attending Mal-Eruption of Wisdom Teeth.*
The numerals in brackets give the numbers of cases in which the symptom or condition was cardinal.
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the analysis has but little statistical value; in both series the figures only record the information supplied, and uniformity of presentation is not to be expected from reporters and authors ranging over the whole world and of widely differing standards. Nevertheless, the positive information obtained has value in showing that certain symptoms and lesions undoubtedly presented themselves and were considered worthy of record. The fact that they may have appeared in other cases without being recorded does not detract from the impressiveness of the tale of suffering which the gathered reports reveal. These records show that in 75 of the 622 cases the symptoms had been erroneously attributed to some cause other than mal-eruption of a wisdom tooth, and in 124 cases ineffective treatment of secondary lesions or disabilities—sometimes prolonged—had been tried before the peccant tooth was extracted and relief obtained. There were post-operative complications in 121 cases, and 42 patients died. The age-range is wide; both the private records and the published cases start at 14 years and end at 84. The commonest period for symptoms to make their appearance seems to be the third decade of life, but 78 of the recorded patients were 50 or over. Several had even appeared to be edentulous for many years before their wisdom teeth, buried beneath the gum, brought the sufferers under observation.

Table I gives the frequencies with which the single teeth, or combinations of them, are noted as being affected. The totals for each sex are remarkably close in the case of both series. For the two series together, the mandibular teeth only (whether singly or together) occur in 456 out of 620* cases, and the upper teeth alone in 123 out of the same total. It may be noted, too, that the frequencies for single teeth are far higher than those for two or more together, though the former are likely to have been favoured by the fact that some of the records are probably incomplete owing to omission of teeth affected at other times. A few frequencies obtained from those in Fig. 1 are discussed in this section without regard to the particular tooth, or teeth, affected, but it is clear from Table I that these relate chiefly to cases having the lower jaw only affected †.

The symptoms fall into several groups, but there are two of these which are larger than any of the others. The first contains those attributable to pressure of the wisdom tooth, especially during the period of eruptive activity. This may make itself felt through local nerves, or the tooth may actually impinge on the second molar and irritate its pulp, causing toothache, or neuralgia, or some other reflex irritation. The second group is the large one associated with pronounced and sometimes fatal septic infection. This may again be subdivided according to whether the symptoms appear spontaneously or follow an attempt at extraction.

* Omitting from the total the two cases for which the positions of the single teeth affected are not given.

† It is hoped that a more adequate statistical treatment of clinical and anatomical data relating to third molars will be published soon. A considerable amount of new material has been collected for this purpose.

TABLE I.

Frequencies of Single Third Molars and Combinations of them affected, compiled from Records of 622 Cases.

Formula of affected molars*	311 contributed cases			311 literature cases			Total 622 cases
	♂	♀	⚥§	♂	♀	⚥§	♂+♀
$\frac{8}{}$	18	8	3	9	15	6	59
$\frac{8}{}$	3	10	1	7	10	2	33
$\frac{8}{}$	40	35	3	42	37	5	162
$\frac{8}{}$	46	37	6	53	34	2	178
$\frac{8}{8}$	3	3	—	4	3	1	14
$\frac{8}{8}$	23	21	1	6	7	1	59
$\frac{8}{8}$	—	3	—	—	2	—	5
$\frac{8}{8}$	—	3	—	2	2	—	7
$\frac{8}{8}$	—	1	—	—	—	—	1
$\frac{8}{8}$	—	1	—	1	1	—	3
$\frac{8}{8}$	1	3	—	—	—	—	4
$\frac{8}{8}$	—	1	—	—	—	—	1
$\frac{8}{8}$	4	7	—	2	7	—	20
$\frac{8}{}$†	3	2	—	4	7	1	17
$\frac{}{8}$†	3	12	5	16	14	7	57
8‡	1	—	—	—	—	1	2
Totals	145	147	19	146	139	26	622

* $\frac{8}{}$ signifies the upper *right* third molar, and so on. There are zero frequencies for the combinations $\frac{8}{8}$ and $\frac{8}{8}$.

† Unspecified as to side, but single tooth affected.

‡ Unspecified as to upper or lower jaw and side, but single tooth affected.

§ Sex not given.

French writers, who have devoted more attention to describing these complications than observers of other countries, divide the septic complications into mucosal and bony. The mucosal group embraces all inflammations of the gum and surrounding mucous membrane, often with extension to the tonsil, throat, lymph glands and salivary glands. These inflammations may be trivial, serious, or fatal. The bony group includes cyst formation, periostitis, and osteomyelitis. Such

conditions appear to be less common than the mucosal, but a far greater proportion of them seem to constitute severe or dangerous illnesses.

Examples of the types of lesions or complications may be given:

(*a*) *Odontalgia, Neuralgia and Reflex Nervous Symptoms.* When the wisdom tooth is erupted but impacted against the second molar, it often becomes a source of trouble, partly through pressure and partly because there is stagnation of food and subsequent decay of the anterior tooth, leading both to odontalgia and to referred neuralgia (Plate I A). When the wisdom tooth is similarly impacted below the gum, it can absorb its way into the anterior molar, even exposing the nerve and giving rise to severe paroxysms of neuralgia. Proximity to the inferior dental nerve, a branch of the trigeminus, is also responsible for much of the severe pain and anomalous reflex nervous symptoms to which this molar gives rise, both during its eruptive activity and when it, or its socket after extraction, has become infected (Plate I B). So close is the mandibular nerve to the root of the tooth (Plate I, B and C) that not infrequently the nerve-trunk itself is crushed or broken during extraction, with resulting numbness of the corresponding half of the lower lip. Occasionally the lingual nerve, which branches from the main trunk before it enters the mandible, is also severed and there is hemianaesthesia of the tongue. In the total of 622 cases local neuralgia occurred in 115, and referred neuralgia in 141 cases. Severe headache, including migraine, was present in 45 instances. There were 32 cases of nervous breakdown, 15 cases of mental depression, and 28 cases of mental derangement. There were also 7 cases of epilepsy, which were arrested and apparently cured by the removal of the impacted tooth.

(*b*) *Sepsis.* Local or general septic complications arise far more frequently from the mandibular than from the maxillary third molar. An upper wisdom tooth is usually either completely erupted or completely buried, and it does not suffer the special risks of partial emergence peculiar to the mandibular tooth. A lower wisdom tooth may erupt normally from the bone in a natural position, but yet be left with its posterior cusps covered by the reflection of mucous membrane which leads from the gum to the ascending ramus (Plate II, A and B). The mucous membrane thus forms a hood over a portion of the erupted tooth—a hood which French writers describe as the *capuchon*—beneath which infection can track quite easily along the biting surface of the tooth. Once bacteria find their way under the hood they are in an ideal position for development and sooner or later a focus of infection inevitably follows. From insignificant beginnings the developments of this complication of the mucosal hood include most severe phlegmonous and anginal infections of the mouth and throat, spreading sometimes to the neck (Plate III A), and even to the mediastinum, or to the cranial cavity, and resulting not infrequently in dangerous illness or death (Plate III B). In the group of septic infections acute suppuration occurred in 134 cases and chronic suppuration in 88 cases. There was osteomyelitis of the jaw in 41 cases, and either extensive cellulitis or Ludwig's angina supervened in 48 cases. Thrombophlebitis of the cavernous sinus occurred in 8, and œdema of the glottis in 6 patients.

(*c*) *Fracture of the Mandible.* The mandible seems to be inherently weak in the region occupied by the wisdom tooth, which is a common site for fracture (Plate IV). When the wisdom tooth is not erupted—i.e. either when it is still in a formative stage within its crypt (Plate V), or when, fully formed, it is retained within the substance of the bone (Plate IV C)—the strength of the bone is further reduced. Fracture may be due to a blow on the side of the face (Plates IV B and V B), or on the front of the chin (Plate IV C), or to misjudged operative force during extraction of the tooth itself (Plate VI). Pathological or spontaneous fracture (Plate VII), succeeding infection of the crypt or of the socket, also occurs and sometimes leads to gross deformity and disfigurement. These cases are due to sub-acute osteomyelitis occurring in the mandible owing to the spread of infection. Necrosis of a portion of the jaw takes place and some inadvertent movement during mastication or sleep causes the jaw to break at the site. Unopposed muscle-pull drags the disjoined portions of the ramus and the corpus apart and the chin swings over towards the affected side. Suppuration, which continues until the necrotic sequestrum is shed, increases the deformity by cicatrising the overlying soft tissues. Fracture was recorded in 48 cases*.

(*d*) *Cystic Growth.* Cystic enlargement of the follicle of the molar, when totally buried, is not uncommon and leads to considerable destruction of the jaw (Plate VIII A). More rarely, also, paramolar dental cysts are found in relation to its roots. Owing to a tendency to spread along the medulla of the bone instead of expanding its inner and outer plates, and because of the total lack of painful symptoms, cysts in this region are often discovered only when they have reached considerable proportions.

A short description of the human dentition may be given before turning to the special question of the eruption of the lower third molars. In man the complete adult dentition consists of thirty-two teeth forming an upper, or maxillary, set and a lower, or mandibular, set—each set consisting of sixteen teeth. The maxillary and mandibular teeth are arranged in the form of two roughly symmetrical arches opposing one another, the shape of which shows considerable intra- and inter-racial variation. The morsal edges of the teeth of the maxillary series are ranged along a curve of larger dimensions than that of the mandibular series, so that the upper incisors pass in front of the corresponding lower teeth and the external cusps of the premolars and molars close somewhat outside those of the mandibular set. In well-developed jaws with normally disposed teeth the articulation of the superior cheek-teeth with the inferior ones is that of a perfect interdigitation of their cusps. In primitive types the third molars have ample room to range with the other teeth and there is often space behind them, resulting in the upper jaw in well-formed maxillary tuberosities, and in the lower jaw in good clearance between the last tooth and the ascending ramus. In types exhibiting lesser development of the jaws the tuberosities are smaller and, while there may

* A photograph of two ancient Egyptian mandibles with healed fractures in the region of the right third molar is given in the present volume of *Biometrika*, Plate II, following p. 172.

be no gross irregularity of the anterior teeth, there is restriction of space posteriorly. Full eruption of the third molars, which are the last teeth to be formed, and their concomitant rotation from their developmental obliquity into a completely vertical position appears to be prevented. The upper wisdom teeth consequently face backwards and outwards, and the lower ones have a corresponding inclination forwards and inwards. This is typical of both the Anglo-Saxon and modern British arches, which are narrow in front with diverging limbs posteriorly. If posterior development of the dental arch is further restricted, we may suppose that the diminution of space available for the molars results in the wisdom teeth remaining partially or wholly buried and impaction is more likely to occur. In these circumstances the mal-erupted teeth may remain more or less in their developmental positions, or they may become grossly deflected by the force of eruption.

The molar teeth vary as to size, and the second and third molars especially vary as to shape, but the variations, either of size or shape, appear to have little or no relation to the size of the jaws. There appears to be no concomitant reduction in size of the molars when the jaws are under-grown. On the other hand, there appears to be some ipsilateral relation in growth between the maxilla and the mandible, for it is commonly found that diminution in size of the superior third molar, or its congenital absence, is accompanied by impaction of the corresponding mandibular wisdom tooth when this is present and of normal dimensions (Plate VIII B). When there is an exceptional difference in size of the upper third molar, compared with the molars anterior to it, this tooth is smaller than the others. Such reduction in size is observed far less frequently in the case of the lower third molar. Also, the upper tooth is not confined by any bony structure posterior to it, as the lower third molar is by the ascending ramus of the mandible. For these reasons more discomfort is associated with eruptive activity of the latter than of the former tooth.

When mal-erupted, the mandibular third molar may occupy any position from one of partial eruption in the normal vertical plane, with three-quarters or more of its crown projecting through the gum, as previously described, to total inclusion deeply within the substance of the bone. Whether partially erupted or completely submerged, it may be found occupying a wide range of attitudes in different jaws (Plate II C). In the human subject, as in the anthropoid apes and New-World monkeys, the infant dentition consists on each side of the mandible of two incisors, a canine and two molariform teeth. The two sides together form the mandibular arcade of the milk dentition which is completed by eruption of the second milk molar, dpm_2*, at two years of age in man (Tomes)†. With the approach towards adolescence these deciduous teeth are shed at intervals and are ultimately replaced by corresponding teeth of the permanent dentition, with the exception that those which succeed the milk molars (dpm_1 and dpm_2) are less molariform in shape and are designated premolars (pm_1 and pm_2). The three lower molars of the per-

* The symbols used in the text to designate the different teeth are those of comparative anatomists, viz.: dpm_1 = 1st lower deciduous molar, pm_1 = 1st lower premolar, M_1 = 1st lower permanent molar, and so on.

† *A Manual of Dental Anatomy, Human and Comparative.* By Sir Charles S. Tomes. Eighth edition edited by H. W. Marett Tims, with the assistance of C. Bowdler Henry, 1923. See p. 227.

manent dentition are developed in the root of the coronoid margin of the ascending ramus and have no deciduous predecessors. They erupt successively posteriorly to the deciduous molars. Normally M_1 begins to erupt through the gum at about $5\frac{1}{2}$ to 6 years of age, before any of the milk teeth are yet shed; M_2 erupts at 12 years, and M_3 at 18 to 22 years. Occasionally, normal eruption of M_3 occurs as early as 13 years or later than 22 years, but it does not appear to undergo normal movement after skeletal growth has ceased. The times of eruption of the teeth have been recorded for several long series of children, but there seem to be no good statistics relating to the third molars. Confirming earlier work of Röse and others, Cattell* showed that at all ages girls are normally in advance of boys in their dental development, but that after 11 years the difference apparently decreases. The second molars were observed to erupt in girls six months earlier than in boys, but the series was not taken beyond the age of 14 years and 9 months and it does not include the third molars.

Plate IX reproduces four skiagrams from Symington and Rankin's *Atlas* illustrating stages in the development of the jaws at the ages of 3, 10 and 13, and in the adult stage. These show the general stages of molar formation and eruption. It is seen that the large M_1 comes into position behind dpm_2 and occupies a space formed for it behind dpm_2 and in front of the coronoid border of the ascending ramus. With continued development M_2 and M_3 successively erupt into position behind M_1 and M_2 respectively. Skulls of young animals, when arranged in age-series, suggest that space is made for the molar teeth behind the last tooth of the deciduous series by a forward growth of the snout which carries with it the whole of the dental arcade in front. Complete eruption of the molar teeth, and particularly of M_3, may be assumed, therefore, to depend upon sufficient forward growth enlarging the space posterior to the post-lacteon†. Series of skiagrams of the mandibles of primates arranged and mounted progressively according to age from birth to maturity show, further, that all the molar teeth develop in their crypts with their occlusal (or chewing) surfaces facing both forwards and also inwards towards the tongue. Indeed, in the chimpanzee and gorilla M_3 may be seen normally placed with its occlusal surface pointing directly inwards towards the mid-sagittal plane. With further growth these transversely placed molars gradually rotate and eventually erupt in a normal vertical attitude as room for them is provided. If growth of the jaw ceases before the tooth is fully in position its impaction ensues.

Plate X shows the skiagrams (enlarged) of the molar regions of the mandibles of three specimens of Rhesus Macaque‡ exhibiting typical stages in the formation

* Psyche Cattell: "Dentition as a Measure of Maturity," *Harvard Monographs in Education*, 1928, p. 67.

† "Post-lacteon" is a term used by Bolk to indicate the most posterior point on the distal surface of the second deciduous molar. In dental terminology "distal" means away from the mid-incisal point, i.e. away from the mid-point of the dental arch; "mesial" means towards this point.

‡ Specimens A 74.21, A 74.22, A 74.23 in the Odontological Collection, Royal College of Surgeons of England. We are indebted to Sir Frank Colyer, K.B.E., Hon. Curator of the Collection, for permission to radiograph this material.

and normal eruption of M_3. The oblique tilt of the developing M_3 may be seen in the middle skiagram. If growth of the jaw should cease before this tooth were fully in position some degree of impaction would result. The anatomical type of impaction would depend partly upon the stage to which normal eruption had proceeded before arrest of growth occurred, and partly upon subsequent deflection of the tooth after encountering obstruction to its own eruptive movement. Were the tooth to remain in the oblique developmental position shown in the middle skiagrams of Plate X it would constitute the common type of impaction seen in man which is termed *mesio-oblique impaction* (Plate I A). If, however, development were to be arrested after the tooth had rotated to a more vertical position, but before complete eruption had occurred, part of the occlusal surface posteriorly might be found to be covered by a portion of the coronoid border of the ascending ramus. This condition is described as *vertical impaction* or *coronoid impaction** (Plate XIV A). Sometimes the tooth rotates to a vertical attitude but remains totally included within the bone and below the level of the anterior molars. It then constitutes a *depressed vertical impaction*† (Plate XIV D). On the other hand, the combined effect of impingement of M_3 against the posterior aspect of M_2 and the force of eruption may cause the obliquely placed M_3, having met obstruction, to rotate in a reverse direction, so that its occlusal surface comes to lie facing the posterior surface of M_2. In this way *horizontal impaction* appears to be produced (Plate XIV B).

A qualitative examination of the skiagrams of 500 medical and dental students and of 135 children below the age of 14 appeared to suggest that the common types of impaction of the lower wisdom tooth are due rather to insufficient growth of the mandible than to primary mal-position of the tooth in its developmental crypt. Comparative anatomy, especially the study of series of skiagrams of the mandibles of anthropoid apes arranged in order of growth as indicated by the stage of eruption of the teeth, also provides evidence that mal-eruption of M_3 may occur if either growth be insufficient to accommodate molars of normal size, or if the molar teeth, and particularly M_3, be abnormally large in an antero-posterior direction for the space provided.

Since the consequences of harbouring an impacted wisdom tooth may be severe, an early diagnosis of probable impaction, made at a stage when effective prophylactic measures may be taken, becomes of special importance. It does not appear to be possible to correct imminent impaction of the wisdom tooth by orthodontic regulation of the teeth, but cases are seen (like the one shown in

* It is of clinical importance that the soft tissues of the gum reach much further forward than the margin of the bone in the form of a hood, or *capuchon*. M_3 may therefore erupt completely from the bone of its crypt, yet a considerable portion of the tooth may still be covered by gum under which infection is free to enter. This condition is known as "soft-tissue impaction."

† As shown in Plate II C, there are other types of impaction besides those here described, e.g. *disto-oblique impaction* and *inverted impaction*, as well as some more rare cases of extreme displacement. The writers have confined their attention to the three varieties which form by far the greatest proportion of cases of impaction met with in clinical practice. No examples of the rarer conditions were found in the series of skiagrams dealt with in later sections of this paper.

Plate XI A) where the early accidental loss of an anterior molar has resulted in space being provided for the full eruption of the wisdom tooth on the same side. This has erupted into satisfactory occlusion, whereas on the other side of the mandible, no anterior tooth having been lost, the wisdom tooth is impacted. An alternative measure to the removal of an anterior tooth is the enucleation of the third molar itself from its crypt while it is still in a developmental stage. This is carried out by an operation recently demonstrated by one of us*.

Plate XI, B and C, shows the skiagrams of a case treated in this way. There are advantages and disadvantages in both procedures, but either is preferable to allowing impaction to occur. The success of both procedures, however, is dependent upon early diagnosis. The preliminary investigation described in the following pages of this paper was undertaken in the hope of discovering a reliable method of making this diagnosis. Mesial tilting in M_3 may be a transitory stage in the course of normal eruption in one individual, but it may represent permanent impaction in another. Our aim in this enquiry was to find a metrical means of distinguishing between these two conditions, and upon this basis of forecasting in the case of an adolescent person the probability of full-eruption or of impaction.

(2) *The Nature of the Investigation and the Material used.* The investigation here described is concerned with the comparison of radiographs of the molar region of the lower jaws of a number of people of different ages. The object of it is to obtain knowledge of the mode of eruption of the third molar which will make it possible to predict, in the case of an immature individual, whether the tooth will ultimately erupt normally, or whether it will become permanently impacted. The method used, briefly, is to trace the outlines of the three molars and certain margins of the bone directly from the radiographs, to obtain enlargements of these tracings on which certain lines are drawn for the purpose of giving a number of defined measurements of *shape* (indices and angles), and, finally, to treat these measurements by statistical methods. The enlarged tracings are also used to define, by purely geometrical means, certain conditions of impaction or normal eruption of the third molar which correspond, in a general way, to the conditions recognised clinically. The idea of obtaining measurements from X-ray films is not novel, though it does not appear to have been applied previously in an attempt to reach a solution of a dental problem. There are obvious dangers in the process, and the reader may well doubt—as did the writers of this paper in the initial stages of the enquiry—whether it is possible to obtain sufficiently accurate measurements in the way described. It is necessary to pay special attention to the "errors" to which the measurements are necessarily subject owing to the different operations involved in determining them.

The material available consisted primarily of four series of radiographs relating chiefly to dental and medical students at three London hospitals. The films actually used for the purpose of the main enquiry number 187, counting the films for each

* "Prophylactic Enucleation of Lower Wisdom Tooth Follicles." *The Lancet*, April 18, 1936, p. 921.

side of the jaw separately in cases where those for both sides are included. These were actually selected from a much larger number for reasons given in the following section of this paper. The material can be summarised in the following way, age x signifying individuals whose age last birthday was x:

Hospital series	Ages							
	22 and over*		21 and 20		19	18		17
	♂	♀	♂	♀	♂	♂	♀	♂
Royal Dental ...	38	6	17	4	9	17	2	1
Westminster ...	4	—	1	1	2	—	—	—
Guy's "unselected"	—	—	20	—	4	2	—	—
Guy's "selected"...	14	—	27	—	13	3	—	2
Totals	56	6	65	5	28	22	2	3

* Of the 62 films relating to individuals aged 22 and over, 23 relate to individuals aged 22, 36 to individuals with ages between 23 and 29, 1 to an individual aged 31 and 2 to an individual aged 42.

Classifying the 127 *individuals* represented according to whether films of both sides of their jaws, or of one only, were used gives:

	Ages					Totals
	22 and over	21 and 20	19	18	17	
Both sides ...	18	22	10	9	1	60
Right side only	16	14	6	5	1	42
Left side only	10	12	2	1	—	25

It is important to note the ways in which the samples of films for students at different hospitals were collected. The material upon which the present investigation is based was not assembled originally for the purpose of biometric analysis. Its collection was incidental to a more comprehensive, but less critical, survey by one of us (C. B. H.) given in two clinical lectures before the Royal College of Surgeons of England*. An examination of the incidence of impaction and of the absence of the third molar in 6000 skulls, representing numerous races and in several collections, had been made and it was thought desirable to add for comparison a sample of living people. It was hoped, also, that it would be possible to re-examine these individuals at intervals of two or three years, in order to gather additional evidence regarding age changes in the factors considered.

* Arris and Gale Lecture (February 21, 1934), "Aberrant Third Molars and their Menace"; Hunterian Lecture (February 1, 1935), "The Aetiology and Treatment of Impacted Wisdom Teeth."

Medical and dental students, whose ultimate whereabouts could be traced through the appropriate directories, were likely to provide the most suitable samples. Requests were therefore made to the authorities of certain colleges in London, and permission to secure radiographs and other records of all students who cared to co-operate was obtained. Grateful acknowledgment is made to the authorities of these colleges, to the radiologists*, and to the students themselves for their much valued assistance. Material collected in two of the colleges is not dealt with in this enquiry. Four series of radiographs are dealt with here, though in each case a selection of all the films available had to be made for reasons explained below. The series relate to:

(*a*) All the students at the Royal Dental Hospital in December 1933 and January 1934, viz. 138.

(*b*) About half of the students, viz. 25, attending the Westminster Hospital at one particular time.

(*c*) Dental students at Guy's Hospital asked to attend if their wisdom teeth were absent or merely had the cusps showing, or if their wisdom teeth were impacted. This will be called the Guy's Hospital "selected" series and it comprises 33 students.

(*d*) Dental students at Guy's Hospital, excluding those in series *c*, and also excluding those whose four wisdom teeth could be accounted for as being extracted or fully erupted and in good functioning position†. This will be called the Guy's Hospital "unselected" series and it comprises 54 students, the total number of dental students in the School at the time being about 320.

These four series form our primary material. The first may be supposed to have been drawn at random, *b* and *d* are probably biassed to some extent, as it is likely that they include an unusual proportion of individuals having an abnormal condition of one or both lower third molar regions, and the other (*c*) is probably biassed in the same way but to a greater extent.

After intervals ranging from 9 to 33 months, one female and 20 male students of the Royal Dental Hospital sample were radiographed again. Comparisons can be made between measurements obtained from 31 of the paired tracings obtained, 11 of the 21 individuals giving radiographs which are suitable for tracing on one side but not on both sides. Counting an individual twice if his radiographs for bot sides were used, the distributions of ages on the two occasions are:

Ages ...	22 and over	21 and 20	19	18	17
First occasion	6	5	5	14	1
Second occasion	10	11	7	3	—

* Royal Dental Hospital—Dr S. Blackman; Westminster Hospital—Dr W. H. Coldwell and Mr L. J. Godden; Guy's Hospital (Dental College)—Dr H. M. Worth.

† The examination of the students was carried out by two members of the staff, Mr J. A. Carleton Hunter and Mr J. A. Hallam.

In the majority of these 31 cases the films were obtained on both occasions by the same radiologist using the same X-ray machine.

Finally, comparisons can be made between data obtained from radiographs of the right side of the jaw of the same man, aged 26, taken within a month by three radiologists* using the same technique and the same apparatus.

(3) *The Taking of the Radiographs and the Method of obtaining Tracings from them.* If exact comparisons are to be made between the radiographs of different people, then it is clear that the method by which they are obtained must be standardised as precisely as possible. In the present case a lateral, or approximately lateral, skiagraph of the molar region of each side of the lower jaw is required. Three ways of obtaining such images—depending on adjusting the head of the subject, the X-ray tube and the film in different positions relative to one another—have been used. The first aims at obtaining a true lateral projection of both sides of the jaw on the same film, and it corresponds to that employed in taking a *norma lateralis* photograph of a head or skull. The axial ray of the X-ray tube on one side of the subject's head is horizontal and perpendicular to his median plane, while the film, on the other side, is parallel to the median plane of his head. Images of the molar regions of both sides of the mandible are thus superposed on the film, if the bone is perfectly symmetrical with regard to the median plane of the head, and they are generally so confused that no accurate tracings relating to the separate sides could be obtained. There is another objection to radiographs of this kind. In the case of the normal jaw the three molars on one side are not aligned in such a way that they are all the same distance from the median plane of the head: usually the first molars are appreciably nearer to the median plane than the third molars. Projection on to a plane parallel to the median plane will thus give an image which is distorted to some extent.

The second method of obtaining the film, known as the intra-oral, is widely used in the practice of dental surgery. The subject bites on a piece of wood to which a small film is attached and adjustment is made so that the film is in contact with, or very close to, the lingual surfaces of his molars on one side. The X-ray tube is then pointed towards the "centre" of the film and held with its axial ray more or less perpendicular to the film. This has some advantages over the previous method—the image of one side only being obtained and the distortion being negligible—but the image is very restricted in size. The complete outlines of all three molars will be shown, but only a small portion of bone immediately behind the third molar will appear, even when the subject pushes the film so far back in his mouth that it causes him discomfort. The inferior part of the corpus of the mandible below the molars is excluded entirely.

The third method of radiographing the molar regions of the jaws has been described recently†. This is designed, like the previous one, to give skiagraphs of

* We are indebted to Drs W. H. Coldwell and A. A. Allchin and their technical assistants for providing this material.

† Sydney Blackman: "Extra-Oral Radiography," *The British Dental Journal*, Vol. LVI. (1934), pp. 113—120.

the two sides separately*, but the film is not inserted in the mouth. The adjustment made can be seen in Plate III C which is reproduced from the paper cited to which the reader is referred for fuller particulars. The X-ray tube in that photograph is fixed behind the angle of the left side of the subject's lower jaw and it points in such a direction that an image of his right molar region will, on exposure, be thrown on to the film contained in the cassette on which his right cheek rests. In this attitude the three molars on the side radiographed are approximately equidistant from the film, but separated from it by a distance which will be different for different individuals. All the radiographs used were taken by this third method, but with the tube at varying distances (up to 5 feet) away from the head of the subject, instead of being almost in contact with him as shown in Plate III C. This modification was introduced in order to reduce distortion, but an increase in the tube-head distance makes it more difficult to ensure that the angle of projection is such that the most effective result will be obtained. A deviation from this ideal angle in one direction will result in an image of the cervical vertebrae being superposed on that of the molar region (as in Plate XIV C), and a deviation in the opposite direction will cause superposition of the image of the other side of the mandible (as in Plate XIV D). Films showing these defects are unsuitable for tracing, and for those actually used the angles of projection must have varied within narrow limits. As we are only concerned with the lower jaw, it is an advantage to have the images of the upper and lower molars quite distinct, and this result can be obtained by making the subject hold a piece of wood between the bite of his front teeth. All the images obtained will be slightly greater than life-size, and the scale of enlargement—depending on the prominence of the cheek-bones, the thickness of the fleshy cheek and the tube-head distance—will be different for different individuals. For this reason comparisons are only made between measurements of shape (indices and angles), and not between absolute measurements of the tracings.

The method of obtaining enlarged tracings from the films may now be considered. There are two alternatives here, both of which make use of an enlarging camera. The first is to throw an enlarged image of the film on to a sheet of glass, and to trace the lines required on a sheet of tracing paper attached to the glass: the other is to trace the lines directly from the film, supported on an illuminated sheet of glass, and then to trace an enlargement of this small tracing. It was soon found that the second of these methods gave more accurate results than the first. The scale of enlargement was 3·4 diameters on the films and hence slightly more than this on life-size. In future work on the same lines greater enlargement—to 5 diameters say—might be used with advantage.

The lines which were traced from the films can be best seen by comparing Plates XIII A—D and XIV A with Figs. 2—6 respectively, the films being reproduced the same size as the originals and the diagrams corresponding to them

* A separate exposure is given for each side, but it is convenient to show the two images on different parts of the same film, one half being covered by a sheet of lead at each exposure.

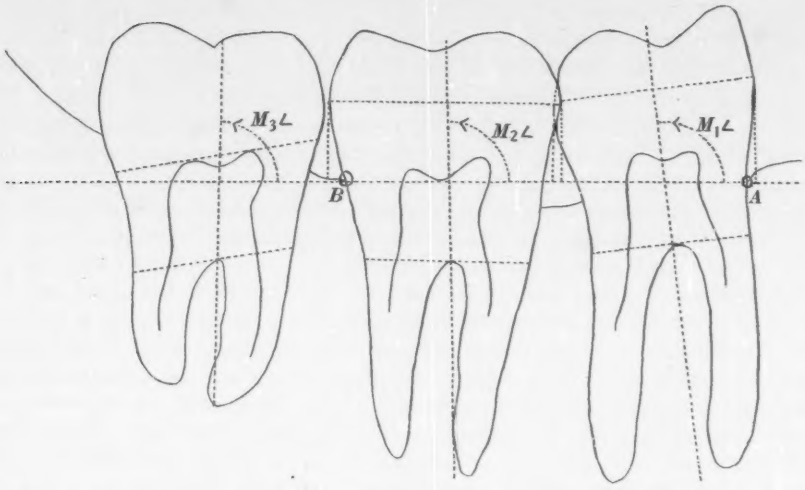


Fig. 2. Showing the base-line (*AB*), the methods of constructing the axes of the molars and the angles between these axes and the base-line. R.D.H. No. 29073. L. ♂. Age 20. Normal.

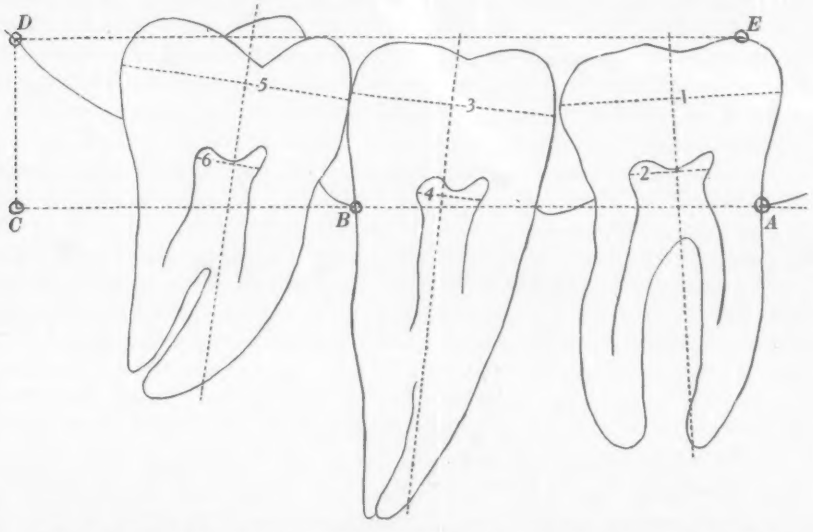


Fig. 3. Showing the method of measuring the space (*BC*) between M_2 and "ramus," and the maximum breadths of the crowns and pulp-cavities of the molars perpendicular to their axes. R.D.H. No. 29204. L. ♂. Age 26. Normal.

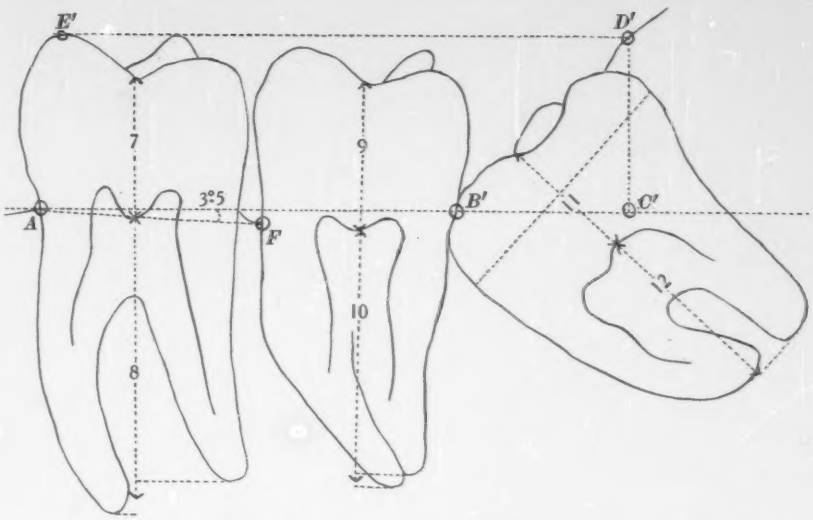


Fig. 4. Showing the arbitrary base-line AB' used when there is no alveolar margin between M_2 and M_3, and the measurements of the "crowns" and "roots" taken along the axes of the molars. R.D.H. No. 29066. R. ♂. Age 18. Mesio-oblique impaction.

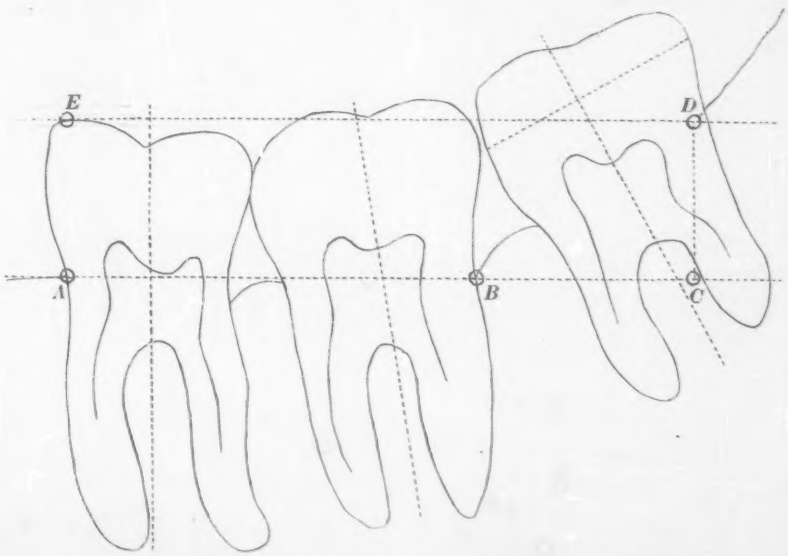


Fig. 5. R.D.H. No. 29226. R. ♂. Age 22. Excessive eruption of M_3.

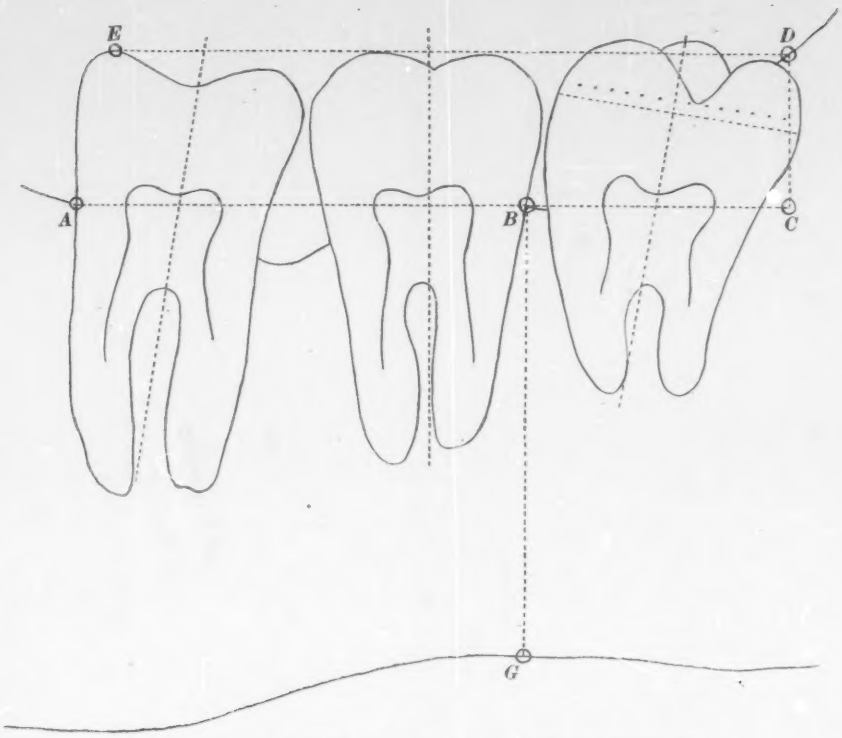


Fig. 6. Showing the method of measuring the height of the corpus (*BG*).
R.D.H. No. 25703. R. ♂. Age 22. Coronoid impaction.

being rather smaller than the enlarged tracings on which measurements were taken. It must be remembered that the plates are not as clearly defined as the original films owing, as is inevitable, to loss in detail due to the processes of reproduction. The outlines of the molars could generally be traced without difficulty. If the outlines of the crowns of two adjacent teeth overlap, it may be possible to see and trace the two intersecting segments (cf. Plate XIII A and Fig. 2), or one of them only may be visible. The projected outline of the occlusal surfaces of a molar often appears as a continuous line only, but above this there may be the outlines of what appear on the films to be cusps which are otherwise hidden. These "hidden" cusps were traced and they are shown in Figs. 3, 4 and 6, though actually they were not used in obtaining measurements from the tracings. Considerable use was made, however, of the continuous lines, particularly that of the first molar, and if these are suspected to be inaccurate owing to artificial filling of the teeth—as in the case of the first and second molars in Plate XIV C, reproducing a radiograph excluded from the series—then no tracing should be taken. If the occlusal surfaces are worn

this will also affect some of the measurements, but in the case of the series used (which relates almost entirely to individuals under 26 years of age) there was no reason to suspect that attrition of the crowns had been marked enough to affect any results appreciably. The outlines of the roots of the teeth can be seen less clearly on many of the films. In the case of the third molars of most of the younger individuals it was clear that the roots were not formed completely, and hence measurements depending on their outlines could be excluded, but for several it was uncertain whether the roots had been completely formed or not, and a few measurements for these may have been included when in fact they should have been excluded. The superior outlines of the pulp-cavities and their continuations towards the roots of the molars are shown on the tracings. These are less definite than the other lines on them, and in a few cases they were so indistinct that they had to be omitted. The alveolar margin is shown anterior to the first molar, between the first and second and, when possible, between the second and third molars. In the case of mesio-oblique impactions—whether temporary or permanent—there may be no alveolar margin between the wisdom tooth and the second molar (Plates XIII C and XIV C and Fig. 4). The inferior border of the corpus of the mandible, below the molars, can be seen on all the films, and it was traced though only shown on Fig. 6. The only other line traced from the films is the one posterior to the third molar. The border of the ramus can generally be seen distinctly on the films. This was used at first, but it was found to give far less effective results than another line which represents the buttress of bone immediately behind the third molar. The location of this line is illustrated by Plate XII which reproduces two photographs and a skiagram of the left ramus of a mandible sawn immediately behind the wisdom tooth. The photograph of this fragment seen from in front (B) shows the anterior rameal border, on the right, forming one side of what appears to be, roughly, a triangular surface having the tip of the coronoid process as its apex and the upper edge of the sawn section as its base. The third side of this triangle represents the buttress in question which is situated immediately behind the third molar. It can be seen on the lateral photograph (A), and the dotted white line connects it with a white band on the skiagram to which it corresponds. This band can be seen on all the radiographs (see Plates I, VI, XI, XIII and XIV), and the upper of the two more or less parallel curved lines bounding it is the one traced and shown on Figs. 2—6.

The essential lines traced from the films are (*a*) the outlines of the three teeth as far down as the point of bifurcation of the roots at least, (*b*) the alveolar margin immediately anterior to the first molar and between the second and third molars—or if there is no alveolar margin between the second and third that between the first and second—and (*c*) the rameal line posterior to the third molar. If any of these lines cannot be seen on the film no tracing is made. It is most desirable that any process used in predicting impaction should be applicable either to all individuals having reached a certain stage of development, or, failing this ideal, to as large a proportion of them as possible. Providing that the radiographs have been taken in a suitable way, the method used in this paper can be applied to all

individuals having the crown of the third molar formed, as long as they have not lost a molar on the side in question and that the occlusal surfaces of the teeth are intact*. Attrition of the molars need not be considered in the case of immature or young adult people, but fillings may make it impossible to trace the outlines of the occlusal surfaces with sufficient accuracy.

The four series of radiographs were obtained before the metrical investigation described in this paper was started. Many of the films were found to be unsuitable for one reason or another. A count was made in the case of the Royal Dental Hospital series, counting the films for each side of the jaw separately and making rejections in the order given, so that not more than one cause is noted for each film although it would have been necessary to reject several of them for more than one reason. The frequencies of causes of rejection obtained in this way are: 7 because part of the outlines of the first molar was not on the film, 136 because of the loss of one or more molars, 1 because the third molar region was obscured by images of the vertebrae, 6 because the first molar region was obscured by the image of the other side of the jaw, 25 because the lines to be traced were too indistinct, 6 because the teeth were clenched, 1 because of a filling. Thus out of a total of 276 skiagrams in the series there were 94 which could be used. It is clear that the loss of a molar was the principal reason for excluding a film as unsuitable, and there is no serious problem of impaction in this case as the early removal of a first or second molar usually allows an impacted third to erupt completely. The other reasons for rejecting films, with the exception of the last, are due to some imperfection in the taking of them which could easily be removed in collecting further material for the same purpose. It may be noted that no films were rejected because of the loss of any tooth, or teeth, anterior to the molars: there are a few such cases referred to in section 6.

(4) *Methods of obtaining Measurements from the enlarged Tracings and the Accuracy of the Measurements.* Figs. 2—6 illustrate methods of obtaining the measurements, the geometrical drawing required being too much to show in a single diagram without confusion. The figures reproduce actual enlarged tracings slightly reduced: Nos. 2 and 3 relate to the *left* side of the lower jaw and the other three to the *right* side. In the transition from the living subject to the final tracing the outlines used may become inverted one or more times, but this does not affect any measurements. On all the figures, A represents the point where the alveolar margin meets the anterior outline of the first molar, and, on all except Fig. 6, B is the point where the alveolar margin meets the posterior outline of the second molar. If there is no line representing the alveolar margin between the second and third molars, as in Fig. 4, then a point B', corresponding to B, is found by a method described below. The line AB (or AB') is used as a base-line to which other lines and angular measurements can be referred.

* There is one other condition which makes it impossible to obtain the measurements used, but only one example of this was found in 578 cases. On this film no trace of the alveolar margin between the first and second molars could be seen, the outlines of these teeth being overlapping.

Fig. 2 shows the methods of constructing the axes of the molars. In the case of the first and second molars perpendiculars to the base-line which are tangential to the anterior and posterior curves of the crowns are drawn first. If part of one of the outlines in question could not be traced owing to the fact that it was obscured in the radiograph by the image of another tooth—as in the case of part of the outline of the first molar in Fig. 2—then the outline is first completed as a smooth curve and the tangent is drawn to this. Such cases of overlapping are quite exceptional. The points of contact of the tangents are joined in the case of each tooth, and parallels to these lines are drawn such that they are tangential to the saddles between the roots of the teeth. Two parallel lines are thus given across the outline of each tooth, and the lines joining their points of bisection are the accepted axes. These axes will always make angles with the base-line which do not diverge much from 90°. It is not necessary to draw any of the tangential lines with great accuracy, since appreciable errors in the directions of the parallel lines will have little effect on the directions of the axes obtained.

The inclination of the third molar shows a much greater range of variation—the tooth being sometimes horizontal—and its axis has to be found without reference to the base-line. The common tangent to the two "horns" of the outline representing the roof of the pulp-cavity is drawn first. This and a line parallel to it which is tangential to the saddle between the roots of the tooth are bisected, and the line joining the points of bisection is the accepted axis of the tooth.

The axes of all three teeth defined in these ways are found in practice to give effective representations of their inclinations to one another and to the base-line. The roots are ignored in defining them and it appears better that they should be. The axes of all the outlines of the molars are shown on every figure, but the lines used in constructing them, which serve no other purpose, are omitted in Figs. 3—6. Angles used ($M_1\angle$, $M_2\angle$ and $M_3\angle$) are measured backwards from the base-line to the axis of the particular tooth, that is to say, they are measured in an anti-clockwise direction in the case of the outlines shown in Figs. 2 and 3, and in a clockwise direction in the case of the reversed outlines (representing the other side of the jaw) shown in Figs. 4, 5 and 6. The first two ($M_1\angle$ and $M_2\angle$) will always be positive and not far removed from 90°: $M_3\angle$ may be negative—in the rare cases of "horizontal" impaction for which the roots are higher than the crown with regard to the base-line—but usually it will have a value between + 45° and + 100°. All these angles could actually be found for all the tracings used, but if any for younger individuals were dealt with in the same way it would become necessary to devise a different method of measuring the inclination of the third molar, since the outlines of the pulp-cavity and of the saddle between the roots would be of different forms or entirely absent.

It was thought that it would be of interest to obtain measures of the maximum breadths of the outlines of the crowns of each molar, and also the maximum breadths of the outlines of their pulp-cavities. The latter measurements are unlikely to be of any value in aiding a solution of the problem discussed in this

paper, but data for them were recorded in the hope that they would prove to be of value later in a comparison between radiographs of the teeth of modern and early prehistoric man. It should be noted that the "breadths" of the outlines of the teeth are really approximately antero-posterior measurements if referred to the mandible. The maximum chords perpendicular to the axis of the particular tooth are the measurements adopted (see Fig. 3), these being found most readily by adjusting a set-square so that it can be moved up and down the outline with one edge remaining perpendicular to the axis and determining the positions of the maximum perpendiculars with the aid of a pair of dividers, which will give the readings required when placed on a scale. There is no difficulty in finding the maximum breadths of the outlines of the crowns in this way: if there is overlapping of the teeth the broken outline of one may be completed as a smooth curve if necessary. It is sometimes more difficult to determine the maximum breadths of the outlines of the pulp-cavities. Normally, the outline in question of any tooth has a "waist" and the maximum perpendicular breadth is found above this. If one point of inflection is higher than the other the measurement *may* be below the higher one but it cannot be below the lower point. Very occasionally there is an inflection on one side, where the "waist" usually comes, but not on the other (see the outline of the pulp-cavity of the third molar in Fig. 4): the breadth is then found at or above this inflection. If there is no indication of the "waist" on either side the measurement is not taken, and cases of this kind were only found for the third molars of some of the younger individuals. The breadth measurements of the teeth are indicated by the numbered broken lines in Fig. 3: the maximum breadth of the crown of the third molar is also shown as a broken line in Figs. 4, 5 and 6, but the other breadth measurements are omitted in these The inter-breadth indices used will be referred to as (i) B, M_1 $(100 \times 2/1)$, (ii) B, M_2 $(100 \times 4/3)$, (iii) B, M_3 $(100 \times 6/5)$ and (iv) max. B, M_3/max. B, M_1 $(100 \times 5/1)$.

There was no special difficulty in devising the measurements defined above, and it is unlikely that any other similar, but differently defined, ones would serve the same purposes much more effectively. The situation with regard to the next measurement is quite different, however. This is intended to express the ratio of the breadth of the third molar to the space between the second molar and the ramus in which the third has to erupt—supposing, for the moment, that only two dimensions are concerned. If the space (or gap) in question is too small compared with the breadth of the tooth for an adult individual then incomplete eruption—i.e. some impacted condition—is to be expected. The maximum breadth of the crown of the third molar perpendicular to the axis of the tooth (5 in Fig. 3) may be supposed a suitable breadth measurement to use for the present purpose. If the *maximum* has passed completely through the alveolar margin then normal eruption may be expected to result. The "best" method of measuring the space between the second molar and the ramus—represented by the rameal line on the tracing—cannot be decided on so easily. If it is represented by x, then, in view of the main purpose of the enquiry, it may be supposed that ideally it should be chosen so that the index $100 \times 5/x$ makes

a maximum separation between the individuals having (*a*) normally erupted and (*b*) impacted third molars, *in the case of those individuals for whom the process of eruption is completed.* It was assumed that this process had been completed for all the individuals represented aged 22 and over, and the distinction between cases of normal eruption and impaction was made by inspection of the tracings in a manner described in the following section of this paper. Having decided these points, an attempt to reach the "best" definition of the space between the second molar and the ramus (x) was made by a somewhat laborious process of trial and error. It is only necessary to describe the best which was found and finally used. Through the highest point with regard to the base-line on the outline of the first molar (E: see Fig. 3) a parallel to the base-line is drawn meeting the rameal line at D. A perpendicular from D on to the continuation of the base-line meets the line at C: the distance BC is the accepted measure of the space between the second molar and the ramus. The index $100 \times 5/BC$ will be called the third molar space index. The lines used in determining BC are also shown on Figs. 4, 5 and 6, E', D' and C' (Fig. 4) corresponding to E, D and C, respectively, and being distinguished because they refer to the arbitrary base-line AB' (defined below) instead of to AB. There is only one difficulty encountered in a few cases: the parallel to the base-line through the highest point on the outline of the first molar may not meet the rameal line if the third molar has continued to erupt until its crown is well above those of the other two teeth (as in Fig. 5). In such a case the rameal line is continued as a smooth curve (always for a short distance) until it meets the parallel. The third molar space index can be found for all the tracings.

Fig. 4 illustrates the method used for measuring the heights of each tooth above and below the pulp-cavity, these being measured along the axes of the teeth already defined. The upper terminal of the upper height is the point where the axis meets the outline of the crown, omitting the outline of what appears to be the far cusp if this is shown. The lower terminal of the lower height is the point on the axis midway between the points where the perpendiculars from the extremities of the roots meet the axis. The upper and lower heights have a common terminal in the point where the axis meets the outline of the pulp-cavity. Indices used express the upper height as a percentage of the lower in the case of each tooth: they will be denoted by (i) H, M_1 $(100 \times 7/8)$, (ii) H, M_2 $(100 \times 9/10)$ and (iii) H, M_3 $(100 \times 11/12)$.

Fig. 4 also illustrates the method of constructing the arbitrary base-line AB' in cases where there is no trace of the alveolar margin between the second and third molars. There are 60 such cases out of the total 187, 59 being classed as examples of mesio-oblique impaction and the other one as an example of coronoid impaction. Suppose, for a moment, that Fig. 4 reproduces a normal tracing, and that B' is the point B for a normal tracing where the alveolar margin meets the posterior outline of the second molar; F being the point where the alveolar margin meets the anterior outline of the second molar. The angle FAB—which will be called the alveolar margin angle—was measured on the 127 tracings on which B

can be found directly, giving a mean of 3°·3 ± ·22* and a standard deviation of 3°·66 ± ·15. It will be shown (in section 7 below) that this angle appears to be unaffected by age and that there is no evidence that it is different for individuals with normally erupted and impacted third molars. In the 60 cases for which B was not given directly the arbitrary base-line was drawn to pass through A and to make an angle of 3°·5† with NA—with the same sense as in Fig. 4—the point where it meets the posterior outline of the second molar being B'. The base-line AB' was then used in place of AB for all purposes. The use of an arbitrary base-line found in such a way is far from satisfactory, but it seemed to be necessitated by the fact that no convenient alternative to the base-line AB which would be applicable in all cases could be found. The device actually used may be expected to give a deviation greater than 5° between the arbitrary base-line and that one which could be used if the point B were available once in 6 cases, or a deviation greater than 10° once in 144 cases‡. It may be noted that an "error" in the base-line even as large as 10° will have little influence on the measurements relating to the first two molars, and none whatever on those for the third molar, owing to the ways in which these measurements are defined. The distance BC—and hence the third molar space index—will be affected more appreciably, but it is unlikely that the conclusions reached below are distorted to any but a slight extent owing to the use of the arbitrary base-line. Its application appeared to give reasonable results in all cases except one (R.D.H. No. 25588) for which the alveolar margin between the first and second molar is peculiarly high, leading to an impossible arbitrary base-line: this tracing was not used.

Fig. 6 illustrates the method of obtaining the last index, which is designed to throw light on the question whether there is any association between impaction of the third molar and the height of the corpus of the mandible in the molar region. The line BG is drawn perpendicular to AB, meeting the outline of the inferior margin of the corpus in G: the index $100 \times AB/BG$ will be called the corpus height index. The outline of the inferior margin of the corpus was not shown on some of the earlier tracings, and all those taken before the time when it was invariably drawn were excluded to avoid the possibility of a biassed selection: the index is only available for 109 of the total 187 tracings.

* The symbol ± is used to denote probable errors throughout this paper.

† This mean value was found for the first 80 of the 127 tracings measured. It would have been slightly better to use the 3°·3 given by the total 127 normal tracings, but the difference is so small that it is of no consequence.

‡ We are supposing here that the readings of the alveolar margin angle obtained are absolutely correct values, not influenced by errors of measurement. This assumption cannot possibly be justified, however. An estimate of personal equation due to the processes of drawing and measuring was only obtained in the case of one angle ($M_3 \angle$), and the standard deviation of the differences between first and second readings was found in this case to be 3°·89 ± ·29. This is actually greater than the standard deviation (3°·66 ± ·15) for the single set of alveolar margin angles for different tracings. If errors made in determining the alveolar margin angle are of the same order, on the average, as those made in determining $M_3 \angle$, then the standard deviation of 3°·7 obtained for the former is probably grossly inaccurate and much too large. It is reasonable to conclude that inaccuracies introduced by using a constant alveolar margin angle of 3°·5 to determine the arbitrary base-line are of the same order as those inaccuracies due in all cases to personal equation.

No useful purpose is served by defining measurements, such as those above, unless it can be shown that they can be determined with sufficient accuracy to give valid statistical results. Some estimate of the reliability of each is obviously required. The factors which will be expected to influence the measurements are:

(*a*) the taking of the radiograph: differences in the form of X-ray machine used, and of its position and inclination relative to the head of the subject and to the film, as well as the position of the head of the subject relative to the film, are all probably of importance;

(*b*) the direct tracing of the outlines required from the film;

(*c*) the tracing of the enlarged image of the direct tracing;

(*d*) the geometrical drawing required on the enlarged tracing;

(*e*) the process of measuring.

We are chiefly interested in the cumulative errors in the measurements incident to the total process, and it is not necessary to consider the effect on personal equation of each factor considered separately. It is interesting to note that it seems to be possible, theoretically at least, to ensure that errors due to the third process (*c*) are less than any assigned limit. If the lines of the direct tracing from the film are supposed to have no thickness, then the *relative* errors made in copying their enlarged images will be inversely proportional to the scale of enlargement. The scale actually used ($\times 3{\cdot}4$ diameters) might be increased with advantage. It is to be expected *a priori* that the accuracy of the measurements will depend principally on factors (*a*) and (*b*) above, i.e. on the way in which the films are obtained and on the accuracy with which the first small tracings are taken from the films.

Three sets of data bearing on this question of personal equation are available. The first was obtained by taking 40 of the total 187 films for which measurements had been obtained, and drawing new small tracings for them which were enlarged and measured in the usual way, but quite independently. The differences between the first and second readings in these 40 cases will give estimates of the cumulative errors due to processes (*b*)—(*e*), the first process—the taking of the films—being excluded from consideration. One angle ($M_3\angle$) and three indices (the third molar space index, B, M_3 and H, M_3) were treated, and it may be assumed that the errors for the indices and angles relating to the other teeth will be approximately of the same order as the corresponding measurements for the third molar. The constants in Table II were obtained from the differences of the readings for the 40 tracings, the second reading being subtracted from the first. The maximum differences found are clearly large enough to be disturbing. No one of the mean differences need be considered to differ significantly from zero, however, which suggests that the measurements were obtained in the same ways on the two occasions. The standard deviations of the differences are all undesirably large compared with the ranges given for 62 individuals, aged 22 and over, which provide some appreciation of the variation in these measurements observed. Actually it is necessary to divide this sample, in ways described below, into sub-samples showing appreciably less

variation of the third molar space index and $M_3\angle$, but not of the indices B, M_3 and H, M_3.

The second set of data providing estimates of the errors made in determining the measurements was obtained by taking new radiographs of 21 individuals (giving 31 repeated tracings) whose tracings were included in the original series of 187. These were all obtained at the Royal Dental Hospital, and in the majority of cases the same radiologist, using the same X-ray machine, took both films for a particular individual. The subjects ranged in age from 17 to 23 years when they were radiographed on the first occasion, and the intervals between the first and second occasions ranged from 9 to 33 months. The differences of the measurements obtained from the first and second films for the same individuals were first worked up for the total 31 pairs of tracings, and the results were very disconcerting. The "errors" made in the case of several of the measurements were evidently large enough to make them almost valueless. Both first and second sets

TABLE II.

Constants for Differences of Measurements obtained from the same Forty Radiographs on two Occasions (First Reading—Second Reading).

Measurement	Third molar space index	$M_3\angle$	B, M_3	H, M_3
Maximum difference	−13	−8°	−12·3	−26·0
Mean difference	−0·30 ± ·52	−0°·51 ± ·41	−2·00 ± ·60	+1·40 ± 1·14
Standard deviation of differences ...	4·87 ± ·37	3°·89 ± ·29	5·65 ± ·43	10·66 ± ·80
Range for 62 tracings (age 22 and over)	43—200	−5°·5—+104°·5	27·6—50·5*	41·2—100·0*

* For 60 tracings.

of the 31 radiographs were then re-examined, and 15 were selected because the lines to be traced could be seen more clearly on these pairs of films than on the remaining 16 pairs. The differences of the measurements obtained from the first and second films in the 15 selected cases are treated in Table III. These constants indicate that for every measurement the differences between the first and second readings tended to be considerably smaller for the selected 15 than for the total 31 repeated radiographs. We must conclude that measurements appreciably more accurate than those for the total series of 187 radiographs used in this paper would have been obtained if only the better half of these—i.e. the radiographs on which the lines traced can be seen most clearly—had been used. It seemed advisable, for the purpose of this preliminary enquiry, to use a larger sample than this at the cost of a loss in accuracy, but it will obviously be desirable to insist on obtaining particularly clear radiographs if further research is carried out on the same lines.

The third set of data was obtained from three radiographs of the right side of the jaws of the same man taken within a month by three radiologists. These films

are particularly clear, and the measurements obtained from them agree well in the case of the third molar space index (53, 54, 54), $M_1 \angle$ (92°·5, 92°·5, 91°), $M_2 \angle$ (86°·5, 84°, 86°) and $M_3 \angle$ (74°·5, 75°·5, 75°). Some of the other indices show much greater divergences.

The rather scanty data in Tables II and III may be used to decide two questions. The first concerns the choice of measurements, from all those taken for the 187 radiographs, which may be employed profitably in this enquiry; the second question concerns the choice of measurements which can be recommended for use

TABLE III.

Constants for Differences of Measurements obtained from Fifteen repeated Radiographs (First Reading—Second Reading).

Measurement	No age change suspected				
	$M_1 \angle$	$M_2 \angle$	B, M_1	B, M_3	Max. B, M_3 / Max. B, M_1
Maximum difference	−9°	−8°	−13·2	−5·5*	+9
Mean difference	−0°·90 ± ·83	−0°·87 ± ·71	−2·92 ± ·96	−2·23 ± ·43*	+2·93 ± ·78
Standard deviation of differences ...	4°·79 ± ·59	4°·08 ± ·50	5·49 ± ·68	2·30 ± ·30*	4·50 ± ·55
Range for 62 tracings (age 22 and over)	64°—100°	59°—97°·5	27·2—50·0‡	27·6—50·5‡	84—127

Measurement	No age change suspected	Subject to age change			
	H, M_1	Third molar space index	$M_3 \angle$	H, M_3	Corpus height index
Maximum difference	+11·8	+23	−12°·5	+27·5†	+11·8*
Mean difference	+2·73 ± 1·02	+6·27 ± 1·95	−2°·40 ± 1·28	+4·18 ± 2·68†	+1·70 ± ·79*
Standard deviation of differences ...	5·86 ± ·72	11·18 ± 1·38	7°·33 ± ·90	13·75 ± 1·89†	4·21 ± ·56*
Range for 62 tracings (age 22 and over)	27·2—64·8§	43—200	−5°·5—+104°·5	41·2—100·0‡	57—97‖

* For 13 pairs of tracings. † For 12 pairs of tracings. ‡ For 60 tracings. § For 61 tracings. ‖ For 31 tracings.

in future enquiries if greater care is taken to exclude radiographs which are not sufficiently clear. It must be remembered that Table II relates to "errors" due to tracing and measuring only, while Table III (based on a selection of the clearer radiographs) relates also to the taking of them. Since intervals of different lengths lapsed between the taking of the first and second sets of radiographs for the same individuals, it is necessary to divide the measurements in Table III into two groups, the first consisting of those believed to be unchanged by growth for the range of ages considered, and the second of those believed to be subject to age changes. A comparison of the data for different age-groups suggests that for ages between 17 and 22 the third molar space, corpus height and H, M_3 indices decrease

with age, while $M_3\angle$ increases. The signs of the maximum and mean differences for these measurements in Table III are thus those which would have been expected, and it must be remembered that in the case of these four the standard deviations of the differences are dependent on age changes as well as on "errors" of measurement. The height indices of the teeth (H, M_1 etc.) appear to be the most unreliable measurements, and it is evident that they should be discarded completely. The breadth indices involving a breadth of a pulp-cavity (B, M_1 etc.) are almost as bad, and it will be safest to discard them also. It seems to be possible to obtain rather more accurate readings of the remaining measurements, but the third molar space index is the only one for which "errors" of personal equation are small compared with the range of values found. Even for this the "errors" may influence results appreciably owing to the fact that the index changes with age, and hence it is necessary to divide the total sample into sub-groups showing lesser variation. It was decided to make no use of the six indices of the type H, M_1 and B, M_1, and, unless the technique can be greatly improved, it would be unprofitable to determine these in future enquiries on the same lines. The hope of obtaining measurements from skiagrams which might be used in investigating taurodontism is thus discouraged. There is really insufficient justification for treating the measurements $M_1\angle$, $M_2\angle$, $M_3\angle$, max. B, M_3/max. B, M_1 and the corpus height index obtained for the 187 tracings, but some statistical comparisons for them are made in section 7 below. If this shows that they promise to be useful, and if it can be shown that they can be determined rather more accurately than in this preliminary enquiry, then their continued use may be recommended. The third molar space index appears to be the most reliable of the measurements defined, but "errors" made in determining it may still be of importance if it is necessary to use sub-samples showing little variation in this index.

(5) *Definitions of Normal Eruption and Classes of Impaction of the Third Molar.* For the purpose of the present enquiry, there appeared to be an obvious advantage in adopting definitions of the conditions* of the third molar which depend only on geometrical relationships between certain lines originally traced, or afterwards drawn, on the enlarged tracings of the radiographs. These definitions are designed to correspond, in a general way, with the definitions used clinically, which are largely based on a visual examination of the radiographs. A clinical verdict may depend partly, however, on the direct examination of soft tissues, and this factor is neglected entirely in applying the geometrical method. The only factors considered in the latter case are the inclination and position of the third molar relative to the other teeth and neighbouring margins of the mandible. An examination of the enlarged tracings for individuals of different ages makes it perfectly clear that impaction in the case of many fully adult individuals is due to the arrest, before completion, of a process of growth which otherwise is normal. When development is arrested the impacted third molar may be retained in

* By "condition" of the third molar, here and elsewhere in the text, we mean the position and orientation of the third molar relative to the two anterior molars and margins of the mandible.

a position relative to the other *teeth* which is precisely similar to that occupied at one stage of development by the tooth of another individual erupting normally and ultimately to reach a normal and completely erupted position (as in the case of most mesio-oblique impactions). Or, less frequently, it may assume a position which is quite different from any assumed by a normally erupting tooth at any stage (as in the case of horizontal and coronoid impactions). Owing to the first of these possibilities, we must expect to find that geometrically defined classes of impaction will give different frequencies for different age-groups, because they will have a different significance according as they relate to individuals for whom the process of eruption has gone as far as it is ever going, or to individuals for whom the process is not completed. As a matter of convenience, we may speak of cases of "permanent impaction" in the first instance and "temporary impaction" in the second: these *may* be placed in the same class if considered without regard to age. Considered only with reference to the vertical plane represented by the tracings, the process of eruption of the third molar can be described in the following way. At age 17—i.e. in the 18th year of life—the tooth is behind and below the second molar, the two not being in contact, and its axis is inclined at an angle of about 45° to that of the second molar, while generally the roots are not fully formed. In the following year the formation of the roots is almost completed, and the third molar moves in the direction of its axis until a point (or surface) on its crown appears to be in contact with a point (or surface) on the posterior face of the second molar slightly above the level of the alveolar margin between the first and second molars. In the next stage of normal development the tooth appears to rotate and slide up the posterior surface of the second molar until it finally reaches a position (about age 22) with its axis approximately vertical and its crown at approximately the same height as those of the first and second molars, while the rameal line appears to meet the posterior surface of the third molar at about the same height as the top of its pulp-cavity. The alternatives to the last stage of this normal process are: (*a*) that the upward progress of the third molar is arrested at some stage after it first appears to be in contact with the second molar, this arrest being permanent; (*b*) that the tooth rotates in the opposite direction until it assumes a permanently impacted position with its crown in contact with the posterior surface of the second molar, and, in the extreme case, its axis approximately horizontal; (*c*) that the third molar rotates completely, and in the usual direction, but that the rameal line appears to meet the crown instead of the posterior surface, so that the tooth does not stand clear, as it were, of the ramus (coronoid impaction); and (*d*) (a condition not generally distinguished) that the third molar assumes a position which is in all respects similar to that of the normally erupted tooth, except that its crown is appreciably higher than those of the first and second molars. It is important to appreciate that, in the case of individuals for whom the process of eruption has gone as far as it is ever going, it is not possible to make any clear distinction between some pairs of these conditions, or between some of them and normal eruption. A continuous series of intermediate stages can be found and any divisions between the classes must necessarily be of

an arbitrary nature. Furthermore, it is not possible to tell in all cases from the condition of the third molar alone whether the process of eruption has gone as far as it is ever going or not: a tooth which is actually impacted permanently with the second molar may appear to be in the same position with regard to that tooth as another which will ultimately assume a normal position.

The classes adopted may now be defined with the aid of Figs. 2—6, all the geometrical drawing required having been described already in defining the measurements. They are:

A. *Mesio-oblique or horizontal impaction* (Fig. 4). All cases in which the third molar is sloping forwards ($M_3\angle < 90°$) and which have any part of the diameter of maximum breadth of the tooth lower than the highest point on the outline of the roof of the pulp-cavity of the second molar are assigned to this class. Generally the third molar will be inclined at an angle close to that of the tooth in Fig. 4, but in extreme cases the axis of the tooth may be approximately horizontal.

B. *Coronoid impaction* (Fig. 6). All cases, excluding those in class A, for which the rameal line meets the outline of the occlusal surface of the third molar. The outline of this surface is defined to be that part of the section of the tooth which lies above the breadth perpendicular to the axis which is tangential to the curve representing the region where the anterior and posterior cusps of the tooth join. This breadth is shown as a dotted line on Fig. 6, and it appears to be invariably above the maximum breadth.

C. *Excessive eruption* (Fig. 5). All cases for which the point of intersection of the axis and diameter of maximum breadth of the third molar is above the parallel to the base-line through the highest point on the outline of the first molar (*ED*).

D. *Normal completed eruption* (Fig. 3). All cases which are not assigned to any one of the above classes.

The definitions of some of these classes may appear to be particularly arbitrary. They were actually chosen, after several trials of other alternative definitions, because they appeared to give the most suggestive separation of the types of condition of the third molar when certain of the measurements are considered. It will be shown below that a solution of the clinical problem does not involve acceptance of the partitioning used here in discussing the nature of the material: it depends, rather, on a method of selecting the most extreme of the cases which fall in classes A and B. For reasons given above, it is to be expected that the proportions of cases falling into the different classes will be different for different age-groups. The frequencies found in the case of the 128 tracings of the total "unselected" series are:

Ages	(*a*) Mesio-oblique impaction	(*b*) Coronoid impaction	(*c*) Excessive eruption	(*d*) Normal	Totals
22 and over	14 (29·2 %)	6 (12·5 %)	4 (8·3 %)	24 (50·0 %)	48
21 and 20	26 (60·5 %)	6 (14·0 %)	—	11 (25·6 %)	43
19, 18 and 17	26 (70·3 %)	5 (13·5 %)	—	6 (16·2 %)	37

For these age-groups there is a marked decrease in the percentage of mesio-oblique impactions and a marked increase in the percentage classed as normal with increasing age. The relative frequency of cases exhibiting coronoid impaction remains practically constant however.

(6) *The Third Molar Space Index.* When using the measurements defined above for any statistical purposes, it must be constantly remembered that the material available is peculiarly heterogeneous. There is diversity on account of differences in age, sex, side, series—the skiagrams for different series having been taken by different radiologists and selected in different ways—in the loss and crowding of teeth other than molars, and in the condition of the third molar. Our main purpose is to discover relationships between the last factor and the measurements, but before this can be done it is necessary to examine the relationships between the other factors and the measurements. The most direct method of carrying out this investigation would be to divide the total material into appropriate sub-groups in order that comparison could be made between one particular factor and the measurement in question, while all the other factors could be considered constant owing to the way in which the sub-groups had been selected. For example, to examine whether the sexes should be differentiated or not in using the third molar space index, we should compare distributions of the index for the two sexes in the case of corresponding pairs of sub-groups such that each pair relates to a single age-group and to the same side, series and condition of the third molar, and then draw deductions from all possible comparisons of this kind. But when such divisions are made the numbers of individuals falling in the different sub-groups are in all cases very small for the present material, and hence it becomes necessary to draw some conclusions which cannot be justified at present with any approach to statistical rigour.

In this section the third molar space index is the only measurement considered. This expresses the maximum breadth of the outline of the third molar as a percentage of the distance between the second molar and the ramus, so that the smaller the index the more space there is for the third molar. Some appreciation of the association between the index and the various factors mentioned above can be obtained from Fig. 7 and similar diagrams not reproduced here. This shows the distributions for all individuals aged 22 and over and for the largest number of sub-divisions possible, classes of the condition of the third molar, series, sexes and sides being kept separate. Similar diagrams were prepared for the age-group 21 and 20 and for the age-group 19—17. They suggest, in the first place, that the corresponding distributions for the right and left sides are not more dissimilar than would be expected owing to random sampling. The best bilateral comparison can be made by considering the differences between the right and left sides in individuals. Indices on both sides are available for 60 people; for 45 of these the conditions of the third molar on the right side and on the left side assign the individual to the same one of the four classes defined in the last section, while for the remaining 15 the right side indicates one class and the left another. The

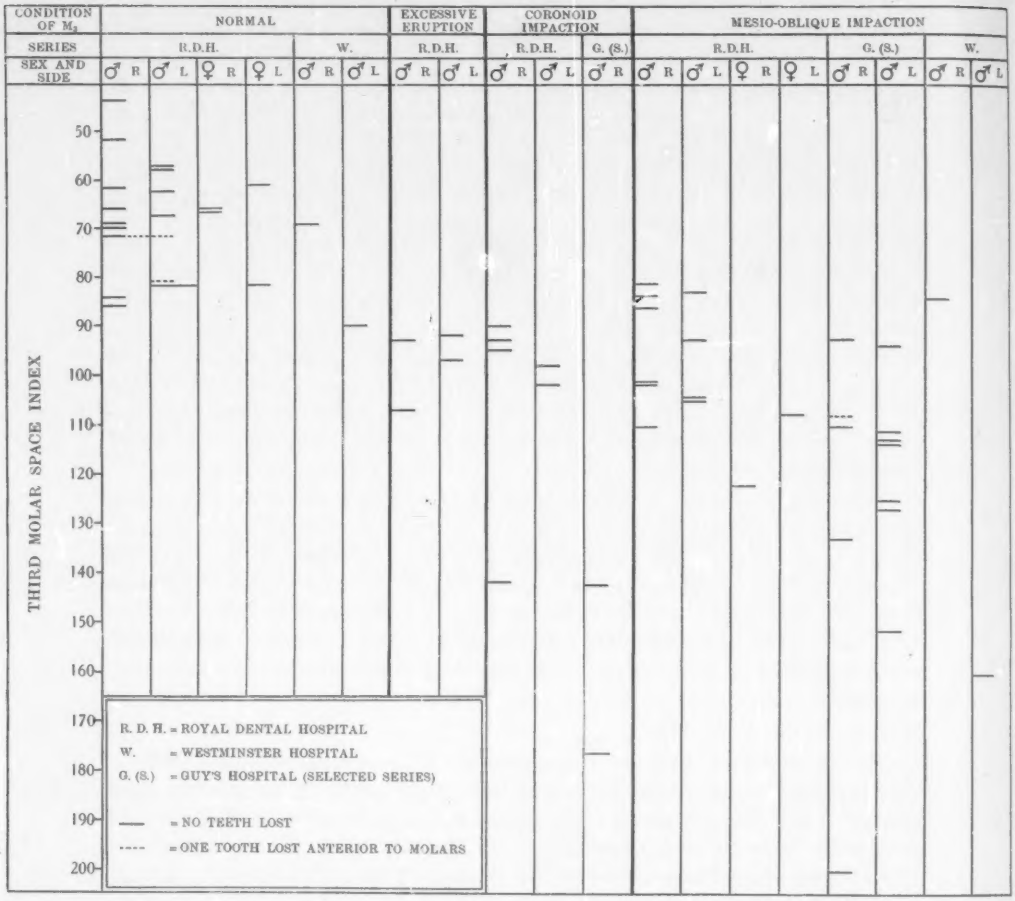


Fig. 7. Distributions of the third molar space index obtained from 62 skiagrams of individuals aged 22 and over.

N.B. The breadth of each column represents two individuals.

bilateral differences (R—L) for the total 60 form a distribution which appears to be approximately normal in form; its range is from -102 to $+92$ with a mean of $+2{\cdot}2 \pm 2{\cdot}7$ and a standard deviation of $31{\cdot}3 \pm 1{\cdot}9$. In spite of the fact that a marked lack of symmetry in the index is frequently found, there is no evidence that it tends, on the average, to be greater on one side than on the other. The bilateral correlation of the indices for the 60 individuals is $+{\cdot}473 \pm {\cdot}068$. This value is decidedly low for a bilateral correlation of a cranial index, but errors of measurement (cf. Table II) may have made it appreciably lower than it should be.

It is evident that the prediction of the index for one side from the known index for the other cannot be made with sufficient accuracy for any practical purposes. Owing to the heterogeneous nature of the material, there can be little justification for comparing all available indices relating to the right side with those available for the left, but it is of some interest to note that the mean for the 102 right is 103·3 and for the 85 left 100·7, while the distributions are markedly asymmetrical. The difference of + 2·6 is only slightly greater than that found when the comparison is restricted to individuals for whom tracings on both sides are available. These data afford sufficient justification for combining readings for the two sides in comparing distributions. A similar bilateral comparison in the case of all the other measurements leads to similar conclusions, and hence readings for all of these were grouped regardless of side. From craniological experience, it would be anticipated that real side differences in averages and variabilities may exist, but that these will probably be so small relatively that it will only be possible to show that they are larger than deviations in the constants due to chance selection when data are available for several hundred individuals. It will be quite safe to assume that bilateral differences of this kind may be neglected in the present preliminary investigation.

The question whether there are appreciable sex differences in the third molar space index cannot be examined so satisfactorily from the available material. Of the total 187 tracings used, 174 relate to men and the remaining 13 to women, while 12 of these 13 are in the Royal Dental Hospital series. Sexual comparisons must hence be confined to this series, and, though differences in side may now be ignored, those dependent on age and the condition of the third molar must be taken into account. There are six female values of the index by the side of male values in corresponding columns in Fig. 7. For cases showing normal eruption of the third molar there appears to be no evidence of sexual differentiation in the index, but for those showing mesio-oblique impaction the two females tend to have larger values than the males in the same group. In every pair of corresponding columns for all ages there are more male than female observations and sexual comparisons can be made in the case of six paired distributions. Subtracting corresponding male and female means, and then weighting each difference according to the number of female readings, leads to a general average divergence (♂ − ♀) of + 6·0. This is probably no larger than differences which are quite likely to arise owing to random sampling, and it will be safest to conclude that there is no evidence that the index shows sexual differentiation. But the data on which this conclusion is based are quite inadequate, and it may well be found for much larger numbers that average values and variabilities of the index are not identical for men and women. Or, if identity is found in the case of mature individuals, there may still be distinction for younger age-groups owing to a sexual difference in the development of the region of the mandible considered in relation to age. While adequate material which could be used to examine such questions is lacking, it was thought that there would be little harm in disregarding sex when dealing with the indices. No clear sexual difference could be found in the case of any of the other

measurements, and accordingly male and female readings were combined in carrying out all the calculations described below.

The question of whether it will be legitimate to pool values of the third molar space index obtained from skiagrams taken by different radiologists may now be considered. Table IV gives the mean values for different series when the different

TABLE IV.

Mean Values of the Third Molar Space Index for different Age-Groups, Series and Conditions of the Third Molar.

Ages	Hospital series	Condition of third molar		
		Normal	Coronoid impaction	Mesio-oblique impaction
22 and over	Royal Dental R.D.	68·9 (22)	103·2 (6)	98·3 (12)
	Westminster W.	79·5 (2)	—	122·0 (2)
	Guy's ("selected")	—	159·5 (2)	123·3 (12)
21 and 20	Royal Dental R.D.	70·1 (8)	91·0 (5)	110·5 (8)
	Westminster W.	63 (1)	—	88 (1)
	Guy's ("unselected") G. (uns.)	67·5 (2)	125 (1)	105·4 (17)
	Guy's ("selected")	116·3 (4)	110·7 (7)	132·7 (16)
19, 18 and 17	Royal Dental R.D.	80·7 (3)	104·0 (5)	104·9 (21)
	Westminster W.	66·5 (2)	—	—
	Guy's ("unselected") G. (uns.)	114 (1)	—	101·4 (5)
	Guy's ("selected")	91·0 (3)	102·0 (2)	111·7 (13)
22 and over	R.D. + W. + G. (uns.)	69·8 (24)	103·2 (6)	101·7 (14)
	Guy's ("selected")	—	159·5 (2)	123·3 (12)
21 and 20	R.D. + W. + G. (uns.)	69·0 (11)	96·7 (6)	106·3 (26)
	Guy's ("selected")	116·3 (4)	110·7 (7)	132·7 (16)
19, 18 and 17	R.D. + W. + G. (uns.)	81·5 (6)	104·0 (5)	104·2 (26)
	Guy's ("selected")	91·0 (3)	102·0 (2)	111·7 (13)

age-groups and conditions of the third molar are kept separate, readings for the right and left sides being combined, and also those for men and women in the same series. Most of the means are still based on very small numbers. The following method was used in attempting to decide whether any pairs of the series may be legitimately combined, or whether they should be treated separately. In the comparison of the Royal Dental Hospital and Westminster Hospital series, for example, we have five pairs of means in the upper part of the table; the difference

is found for each pair, with regard to sign, and the weighted average is found of these five differences by assigning a weight to each which is the smaller of the two numbers on which the means from which the difference is derived are based. In the particular case considered all the Royal Dental Hospital means are based on larger numbers of readings than the corresponding Westminster Hospital means, so the numbers for the latter are used throughout in weighting the differences. The weighted average differences for all pairs of series are given below, the numbers in brackets being the sums of the weights:

Royal Dental – Westminster = – 1·3 (8),
Royal Dental – Guy's "unselected" = – 0·2 (17),
Westminster – Guy's "unselected" = – 23·1 (3),
Royal Dental – Guy's "selected" = - 20·3 (49),
Westminster – Guy's "selected" = – 29·0 (6),
Guy's "unselected" – Guy's "selected" = – 21·9 (25).

The first two of these mean differences are evidently of no account: the third would hence be expected to be insignificant as well and the large value of 23·1 found is based on so few comparisons that no reliance can be placed on it. The means for the Guy's Hospital "selected" series appear to be markedly, and approximately equally, removed from those for the other three series. We need some estimate of the significance of these differences. It is seen from Table IV that the difference between the means (each based on 12 individuals) for the Royal Dental Hospital and Guy's "selected" series is 25·0 for the cases of mesio-oblique impaction and the age-group 22 and over. This difference is found to be 4·2 times its probable error. For all comparisons possible the pooled differences of the means between the Guy's "selected" series, on the one hand, and the Royal Dental and Guy's "unselected" series, on the other, are 20·3 (49) and 21·9 (25). It will be safe to assume that both these values differ from zero with marked significance. The difference of 29·0 (6) for the Westminster and Guy's "selected" series is probably insignificant however. Judging from all these data, it is reasonable to conclude that the mean indices for the three "unselected" series differ insignificantly from one another, while each differs significantly from the means for the Guy's Hospital "selected" series, or would be expected to do so if larger numbers of readings were available. There seems to be sufficient justification for combining values of the index for the first three series, as is done in the lower part of Table IV and in Fig. 8. The differences of these pooled means from those for the Guy's "selected" series are rather erratic, but they tend to be more nearly equal in cases where the means from which they are derived are based on larger numbers. The weighted average difference computed in the way described above is – 20·5 (58). Two objections may be made to the pooling of the different series adopted. The first is that the true differences between the means for two particular series may not be the same for all the groups, so that there is little justification in basing conclusions on a general average obtained from the observed values for all the groups. The second objection is that in any case it is not sufficient to show that the means of two

distributions differ insignificantly, since their forms and variabilities may still be significantly different. The material available is obviously insufficient in bulk to make investigation of either of these points profitable. There still appears to be good justification for combining the series in the way described. In doing so we are accepting the working hypothesis that the indices obtained from the skiagrams taken by the different radiologists are comparable. The divergence of those for the Guy's Hospital "selected" series can be attributed to the fact that the subjects were selected, while all the others were taken more or less at random from the general population.

Granting the assumptions made, we are now in a position to examine the relationships between the third molar space index, age and the condition of the third molar. A rough comparison of the distributions shown in Fig. 8 may be made first, starting with the "unselected" individuals aged 22 and over. Those with the third molars normally erupted show values of the index ranging from 43 to 90; all four showing excessive eruption of the tooth have somewhat higher values (93—107); in cases of coronoid impaction the range is from 90 to 142, and in cases of mesio-oblique impaction from 81 to 160. The normal series thus covers a markedly different range from the other three, though its tail would doubtless overlap all theirs if more observations were available. Of the 24 normal cases there are 17 (71 %) having indices less than those found in any other group. The distributions for cases of excessive eruption of the third molar, and for coronoid and mesio-oblique impactions, are not clearly distinguished from one another. For this age-group, then, the index differentiates the normal and non-normal individuals in a high percentage of cases. Still considering the "unselected" series only, it is found for the age-group 21 and 20 that the normal overlaps the two non-normal distributions far more, and for the youngest age-group the overlap is still more marked. It may be noted, too, that while for any particular age-group the distributions for coronoid and mesio-oblique impactions overlap almost completely, yet the latter tend to have more extreme high values than the former, even allowing for the fact that in most cases they are made up by the larger number of individuals. This is suggestive, since the worst cases of impaction are mesio-oblique rather than coronoid, and it encourages the hope that the index can be taken as a measure of the severity of impaction. If we are on the right lines in making this supposition, then it is easy to explain why the normal and non-normal distributions overlap more in the case of the immature than in the case of the mature individuals. The proportions in each class are quite markedly different for the different age-groups (see table on p. 406). The process of eruption may allow a third molar which is classed as a case of mesio-oblique impaction at age 20, say, to rotate and assume a normal position by age 22. To reconcile the frequencies found for the "unselected" series we can hence suppose, for example, that about half of the cases which exhibit mesio-oblique impaction at age 20 will ultimately become normal, while the other half will remain permanently impacted, though perhaps not until the third molar region has become modified to some extent in such a way as to change the value of the third molar space index. The

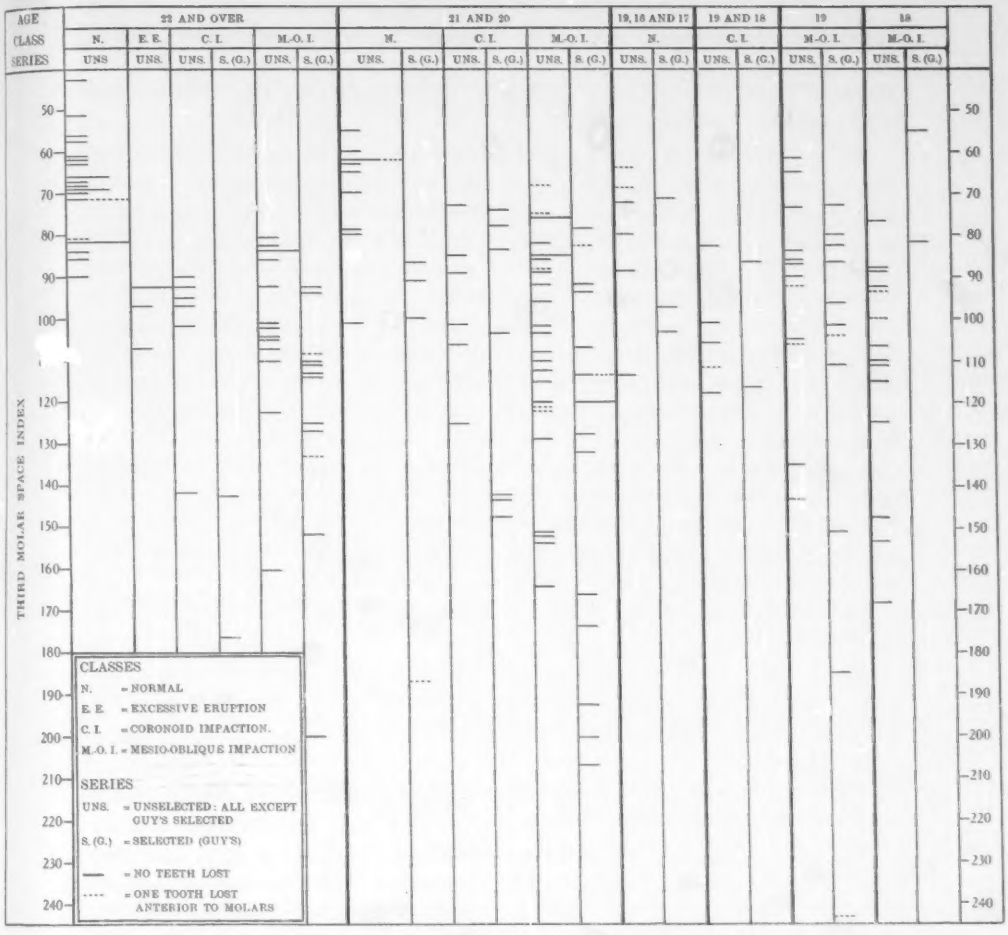


Fig. 8. Distributions of the third molar space index for the total 187 cases.

N.B. The breadth of each broader column represents three individuals, and that of the narrower columns two individuals.

distributions for the impacted classes will hence be expected to be more variable, and to overlap the normal distributions more, in the case of the younger age-groups than of the mature age-group. The following standard deviations are found for the short series exhibiting mesio-oblique impaction of the third molar:

Age-group ...	22 and over	21 and 20	19 and 18
Total "unselected"	20·2 (14)	26·4 (26)	26·9 (26)
Guy's "selected"	27·9 (12)	40·5 (16)	50·1 (13)

The differences between these values are striking: there is a clear decrease in variability with increasing age, and the "selected" groups are much more variable than the "unselected," as would have been anticipated owing to the fact that abnormal cases obviously show much greater variation in the index than normal cases (see Figs. 7 and 8).

Having thus gained some appreciation of the nature of the material, we may now ask whether it may be used to give effective prediction of ultimate impaction from a knowledge of the value of the third molar space index at ages before maturity of the region is reached. The assumption which will be made is that the index actually provides a measure of the severity of impaction, and the problem concerns the way in which it changes with age. If application is to be made to individuals, then the only truly satisfactory estimates of age changes will be those derived from measurements of the same individuals at different ages. Ideally we should have follow-through (seriatim) records, and no others can give results as conclusive as they would. By making certain assumptions, however, we can draw inferences regarding the nature of the age changes from data, such as those available, for which the series in different age-groups relate to different individuals. This requires comparison of the different total distributions, irrespective of the classes of condition of the third molar, for each age-group used, and the problem is to discover how those for the younger ages become transformed into the distribution for mature individuals. The numbers available are small, and it is clear that sampling errors will lessen appreciably the possibility of reaching any exact solution.

We are only concerned now with the total "unselected" series. The actual frequencies for different ranges of the index in the case of each age-group are given in Table V, and constants derived from the distributions are:

Age-group ...	22 and over	21 and 20	19, 18 and 17
Mean	$85{\cdot}6 \pm 2{\cdot}1$ (48)	$95{\cdot}4 \pm 2{\cdot}8$ (43)	$100{\cdot}5 \pm 2{\cdot}7$ (37)
Standard deviation	$22{\cdot}0 \pm 1{\cdot}5$	$27{\cdot}3 \pm 2{\cdot}0$	$24{\cdot}4 \pm 1{\cdot}9$

The three standard deviations do not differ significantly from one another, and only one of the differences of the means is significant: the mean for the youngest age-group exceeds that for the eldest by an amount which is 4·4 times the probable error of the difference. Using the (χ^2, P) method, given by Professor Karl Pearson*, to measure the probability that two independent distributions of frequency are really samples of the same population, we find the following values of P, the numbers in brackets being the numbers of groups used:

(Age 22 and over with 21 and 20)	(22 and over with 19, 18 and 17)	(21 and 20 with 19, 18 and 17)
·334 (7)†	·477 (6)	·693 (6)

These probabilities are all quite high, but we may feel confident that they would be lower if larger samples were available. An attempt was made to discover by

* *Biometrika*, Vol. VIII. pp. 250–254.

† Using the six groups available in the other two cases gives a P of ·314.

TABLE V.

Actual and adjusted Distributions of the Third Molar Space Index: Total "Unselected" Series.

Third molar space index	Type of distribution	Age-group					
		22 and over		21 and 20		19, 18 and 17	
		Frequency	Percentage	Frequency	Percentage	Frequency	Percentage
40—60	Actual	4	8·3	1·5	3·5	0	0
60—80		13	27·1	13	30·2	7·5	20·3
80—100		20	41·7	10·5	24·4	12	32·4
100—120		8	16·7	9·5	22·1	11·5	31·1
120—140		1	2·1	4·5	10·5	2	5·4
140—160		1·5	3·1	3	7·0	3	8·1
160—180		0·5	1·0	1	2·3	1	2·7
Totals		48	100·0	43	100·0	37	100·0
40—60	Adjusted for younger age-groups: "proportional squared reduction"	—	8·3	6	14·0	3	8·1
60—80		—	27·1	14	32·6	13	35·1
80—100		—	41·7	13	30·2	15	40·5
100—120		—	16·7	6	14·0	4	10·8
120—140		—	2·1	4	9·3	2	5·4
140—160		—	3·1	0	0	0	0
160—180		—	1·0	0	0	0	0

inference how one distribution becomes transformed into another owing to growth changes in individuals. The problem is to find a method of adjusting the distributions for the two younger age-groups so that maximum values of P are found when they are compared with the adult distribution. Three methods of doing this were tried, and only one of these gave a suggestive result. The assumptions made in this case are (i) that with increasing age there is invariably a decrease in the value of the index until growth in the molar region of the mandible ceases, (ii) that the mean index for the adjusted distributions for the younger age-groups should be the same as that for the available sample relating to individuals aged 22 and over, and (iii) that the amounts to be subtracted from the individual indices are proportional to the squares of their values. The adjusted distributions for the two younger age-groups found in this way are given in the lower part of Table V and they give the following P's:

(Age 22 and over with adjusted 21 and 20)	(Age 22 and over with adjusted 19, 18 and 17)	(Adjusted 21 and 20 with adjusted 19, 18 and 17)
·722 (5)	·912 (5)	·769 (5)

The first two of these values show a marked improvement on the corresponding P's for the crude distributions, but the method of inferring the nature of the age change is an indirect one and no reliance can be placed on it.

The possible use in practice of the conclusions reached above may now be considered. Of the 48 cases comprising the total "unselected" series for the age-group 22 and over, there are 20 in the coronoid and mesio-oblique impaction classes which have been distinguished by geometrical means. The practical problem is to decide whether it would have been possible to predict ultimate impaction in some, or all, of these 20 cases at some age before maturity of the region had been reached, so that suitable measures for correction or prevention, such as the removal of the wisdom tooth or of a molar anterior to it, might have been taken. The third molar space index will effect this purpose if it can be supposed that a higher index indicates a more severe degree of impaction, and hence more cause for intervention. It is suggested that the evidence discussed above is sufficient to justify this supposition. Accepting it, it might be decided arbitrarily to intervene if the index for an individual aged 22 or over is 120 or greater. If it could be inferred from a knowledge of the index for an immature individual that his index would be greater than 120 on reaching maturity, then intervention would be justified. The material is not sufficient for accurate prediction of this kind to be made at present, but it could be derived from a sufficient number of radiographs for the same individuals at different ages. The need is for series of data relating to 100 persons of each sex, say, each being radiographed at intervals of one year from age 16 to age 22 and for a comparison of the measurements obtained with clinical records.

In the foregoing discussion of the third molar space index no account has been taken of the possible influence on the index of the loss or crowding of teeth anterior to the molars. Among the total 187 cases there are 23 having lost one such tooth, and no cases of more than one having been lost on the side in question. These are indicated in Fig. 8. In the case of the "unselected" series there is no suggestion, judging from all possible comparisons, that the loss of a tooth anterior to the molars tended to give the individual either a higher or lower value than the average for the particular group in which he falls. The same is true for the "selected" series of students at Guy's Hospital with two exceptions. The first of these (No. 19 R, age 20) had the first premolar extracted at 17 years, and the gap between the canine and second premolar was closed by the time the skiagram was taken. Judging from the tracing (in the way described on p. 406 above) the third molar had erupted normally, though it and the first and second molars are sloping forwards to an unusual degree. The index found is 187, which is markedly greater than the next largest (114) for a "normal" individual of any age in either "selected" or "unselected" series. The criterion suggested would indicate that the third molar should be extracted in spite of apparent normal eruption, and this is the only case in the total 187 for which a contradiction of this kind is found. The other case of an individual with a tooth lost having an exceptional index (No. 46 R, age 19)

is obviously one of severe impaction: the central incisor had been extracted at 11 years, and there is crowding in the incisor region. The index of 243 is the highest found, the nearest to it being 207. Judging from all the possible comparisons, it cannot be concluded that the loss of a tooth anterior to the molars affects the index to any appreciable extent, and accordingly no distinction was made between cases with complete lower dentitions and those with one such tooth lost. A fuller investigation of this question with regard to the particular tooth lost and the age at extraction would be of interest, but it would require a far larger number of observations. Crowding of any teeth anterior to the molars was also recorded, and a similar method of comparison failed to suggest that there is any correlation between this condition and values of the index. These conclusions are contrary to expectation.

No account has been taken of errors of measurement in the above discussion of the third molar space index. Taking into account only the processes of drawing and measuring, the standard deviation of the differences between first and second readings for the same series was found to be $4{\cdot}87 \pm {\cdot}37$, the signs of the differences being taken into account, and the largest difference found (among 40) was 13. The "errors" made in measuring are not large enough to have had any appreciable effect on the general conclusions reached, but they are sufficiently large to be disturbing when applications are made to particular individuals. If it is decided that operative treatment is required in the case of individuals having the index greater than a certain limiting value, then all readings found, on a first trial, to lie in the neighbourhood of the limit—diverging from it by less than 10 on either side, say—should be repeated one or more times so that the mean of multiple readings could be used.

(7) *The other Measurements and their Relationships to the Third Molar Space Index.* All the other measurements, except the six rejected, will now be discussed in order to determine whether any of them may be used, in conjunction with the third molar space index, to make the prediction of ultimate impaction more precise, or whether, apart from this question, they throw any light on the growth changes which take place in the molar region of the mandible. It must be remembered that the crude estimates obtained of the kind of "errors" made in determining these measurements have suggested that their readings for the 187 tracings are far from reliable. Comparisons for them similar to those made in the case of the third molar space index suggest that no account need be taken of differences in sex, and that all the "unselected" series may be combined in compiling distributions. The constants for the pooled "unselected" and the "selected" series for different age-groups and different conditions of the third molar must still be compared before further pooling of the material will be justified. There is some justification, however, for neglecting all the other factors when asking whether constants for the right and left sides are differentiated, as long as comparisons are restricted to the 60 individuals for whom measurements on both sides are available. The bilateral constants for this sample are given in Table VI, and all these must be supposed

inaccurate, in varying degrees, owing to "errors" made in determining the measurements. For this reason the standard deviations must be supposed too large, and the bilateral correlations too small. All the differences of means in Table VI differ insignificantly from zero, and there are no clearly significant differences between the corresponding standard deviations of the measurements on the right and left sides. An estimate of the personal equation due to processes of drawing and measuring was obtained for $M_3\angle$ (see Table II), and the standard deviation of the differences between first and second readings was found to be about 4°. The standard deviations of differences between first and second readings obtained from different radiographs of the same individuals were also found to be of the order 4° for $M_1\angle$ and $M_2\angle$. In view of these findings, the bilateral correlations of the angles given in Table VI must be supposed appreciably smaller than the true values for the sample owing to "errors" of measurement, the reduction probably being very

TABLE VI.

Bilateral Comparisons of the Angular Measurements and two Indices.

		$M_1\angle$	$M_2\angle$	$M_3\angle$	Alveolar margin $\angle$	$\frac{100\times \text{Max. } B, M_3}{\text{Max. } B, M_1}$	Corpus height index
Means	*R*	89°·7 ± ·49 (60)	84°·7 ± ·62 (60)	66°·1 ± 2·1 (60)	3°·7 ± ·46 (32)	108·2 ± ·83 (60)	76·9 ± 1·2 (46)
	L	88°·6 ± ·65 (60)	84°·1 ± ·74 (60)	59°·4 ± 2·3 (60)	2°·6 ± ·38 (32)	106·6 ± ·83 (60)	75·8 ± 1·1 (46)
	R − *L*	+1°·1 ± ·62	+0°·6 ± ·59	+6°·7 ± 2·5	+1°·1 ± ·59	+1·6 ± ·89	+1·1 ± ·70
Standard deviations	*R*	5°·65 ± ·35	7°·16 ± ·44	24°·6 ± 1·5	3°·85 ± ·32	9·49 ± ·58	12·48 ± ·88
	L	7°·50 ± ·46	8°·55 ± ·53	26°·1 ± 1·6	3°·19 ± ·27	9·54 ± ·59	10·72 ± ·75
	R − *L*	−1°·85 ± ·52	−1°·39 ± ·53	−1°·5 ± 2·1	+0°·66 ± ·42	−0·05 ± ·75	+1·76 ± ·67
Correlations	*RL*	+·442 ± ·070	+·633 ± ·052	+·305 ± ·079	+·029 ± ·119	+·426 ± ·071	+·818 ± ·033

great in the case of the alveolar margin angle—for which the standard deviations for the right and left sides are actually less than 4°—marked in the case of $M_1\angle$ and $M_2\angle$, and of little account in the case of $M_3\angle$. The true bilateral correlations for $M_1\angle$ and $M_2\angle$ are probably very high, while that for $M_3\angle$ must be appreciably less. The true bilateral correlations for the two indices in Table VI are probably very high. No distinction was made between right and left sides in calculating the constants for these measurements discussed below.

We may now examine the relationships between them with reference to the "unselected" and "selected" series, age and the condition of the third molar. Instead of dividing the total sample into a considerable number of sub-samples with a very small number of individuals in each, as was done in the case of the third molar space index, the method adopted was to consider each factor separately for all the individuals, while ignoring the other factors. The constants in Table VII

TABLE VII.

Mean Values of the Angular Measurements for different Groups.

		All ages and classes		All classes and series			All ages and series				Total
		Series		Ages			Classes				
		Unselected	Selected	22 and over	21 and 20	19, 18 and 17	Normal	Excessive eruption	Coronoid impaction	Mesio-oblique impaction	
$M_1 \angle$	Mean	89°·0 ± ·41 (128)	89°·2 ± ·55 (59)	88°·9 ± ·51 (62)	88°·8 ± ·51 (70)	89°·5 ± ·70 (55)	87°·7 ± ·76 (48)	84°·6 (4)	90°·8 ± ·58 (28)	89°·5 ± ·42 (107)	89°·1 ± ·33 (187)
	σ	6°·84 ± ·29	6°·25 ± ·39	5°·95 ± ·36	6°·30 ± ·36	7°·69 ± ·49	7°·85 ± ·54	—	4°·45 ± ·40	6°·46 ± ·30	6°·66 ± ·23
$M_2 \angle$	Mean	83°·9 ± ·44 (128)	86°·8 ± ·68 (59)	84°·0 ± ·66 (62)	85°·8 ± ·55 (70)	84°·5 ± ·70 (55)	82°·0 ± ·65 (48)	72°·3 (4)	85°·2 ± ·83 (28)	86°·5 ± ·48 (107)	84°·8 ± ·37 (187)
	σ	7°·31 ± ·31	7°·74 ± ·48	7°·74 ± ·47	6°·78 ± ·39	7°·72 ± ·50	6°·70 ± ·46	—	6°·49 ± ·58	7°·35 ± ·34	7°·54 ± ·26
$M_3 \angle$	Mean	63°·0 ± 1·4 (128)	51°·4 ± 2·6 (59)	62°·4 ± 2·3 (62)	56°·9 ± 2·4 (70)	58°·9 ± 2·0 (55)	76°·3 ± ·97 (48)	60°·3 (4)	87°·8 ± 1·5 (28)	44°·3 ± 1·6 (107)	59°·3 ± 1·2 (187)
	σ	23°·7 ± 1·0	29°·8 ± 1·9	26°·9 ± 1·6	30°·1 ± 1·7	21°·9 ± 1·4	10°·0 ± ·69	—	12°·0 ± 1·1	24°·2 ± 1·1	23°·9 ± ·83
Alveolar margin $\angle$	Mean	3°·5 ± ·25 (92)	2°·9 ± ·42 (35)	3°·7 ± ·30 (56)	2°·7 ± ·42 (43)	3°·3 ± ·43 (28)	3°·9 ± ·26 (48)	6°·0 (4)	2°·9 ± ·45 (27)	2°·8 ± ·41 (48)	3°·3 ± ·22 (127)
	σ	3°·59 ± ·18	3°·07 ± ·30	3°·31 ± ·21	4°·06 ± ·30	3°·39 ± ·31	2°·71 ± ·19	—	3°·46 ± ·32	4°·24 ± ·29	3°·66 ± ·15

were found in this way. Such a method of comparison is likely to lead to fallacious conclusions if the heterogeneous nature of the material is forgotten. For example, in comparing the constants for a particular character of the total "unselected" and total "selected" series we are ignoring the fact that the age distributions and the proportions falling in the different classes of the condition of the third molar are different for the "unselected" and "selected" series. But if, for a particular character, no significant differences are found between the means or standard deviations in any one of the divisions of the table, then it will be fairly safe to assume that the character is not influenced appreciably by any of the factors. Counting as a significant difference one which exceeds 3·5 times its probable error, we can summarise the comparisons in the following way, "none" meaning that no significant differences are found between any pairs of constants belonging to the same division:

$M_1\angle$—means, none; standard deviations, for coronoid impactions less than for normal ($\Delta/(\text{p.e. }\Delta) = 5{\cdot}1$) and mesio-oblique impactions (4·0).

$M_2\angle$—means, for "unselected" less than for "selected" series (3·6), and for normal less than for mesio-oblique impactions (5·6); standard deviations, none.

$M_3\angle$—means, for "selected" less than for "unselected" series (3·9), for normal less than for coronoid impactions (6·4), and for mesio-oblique impactions less than for normal (17·1) and coronoid impactions (19·8); standard deviations, for ages 19, 18 and 17 less than for ages 21 and 20 (3·7), and for mesio-oblique impactions greater than for normal (10·9) and coronoid impactions (7·8).

Alveolar margin$\angle$—means, none; standard deviations, for normal less than for mesio-oblique impactions (4·4).

Most of the relationships revealed by this comparison would have been expected. All the most significant differences between means or variabilities are found when different classes of the condition of the third molar are compared, and $M_3\angle$ differentiates the groups more effectively than any one of the other angles, while these show little sign of being affected to more than a slight extent by the eruption of the third molar. Comparisons with the four cases of excessive eruption of the third molar are of some interest: these relate to four individuals aged 22 or over. Their $M_1\angle$'s are not peculiar, but the second molars are all tilted forwards to an unusual extent ($M_2\angle = 59°$, 74°, 75°·5 and 80°·5), and the third molars are tilted as much forward as is normally found in cases of permanent mesio-oblique impaction ($M_3\angle = 55°{\cdot}5$, 64°, 60° and 61°·5, respectively). The distinctiveness of the condition of excessive eruption thus appears to depend on the angular measurements as well as on the third molar space index, though far more material would be needed to investigate it adequately.

The correlations between the angular measurements and with the index are of more interest. These are given in Table VIII for all the cases of normal eruption, for all the cases of mesio-oblique impaction and for the total series, except in the case of the alveolar margin angle for which the total series only was used. The use of such groups, which combine data for all ages, cannot be justified rigorously, but a comparison of results for them may at least be suggestive. As would have been

TABLE VIII.

Correlations of the Third Molar Space Index and the Angular Measurements.

		Third molar space index	M_1 ∠	M_2 ∠	M_3 ∠	Alveolar margin ∠
Third molar space index	Normal	—	−·272±·090 (48)	−·208±·093 (48)	−·350±·085 (48)	—
	Mesio-oblique impaction		−·317±·059 (107)	−·184±·063 (107)	−·311±·059 (107)	—
	Total		−·224±·047 (187)	−·073±·049 (187)	−·437±·040 (187)	−·203±·057 (127)
M_1 ∠	Normal	−·272±·090 (48)	—	+·632±·058 (48)	+·343±·086 (48)	—
	Mesio-oblique impaction	−·317±·059 (107)		+·569±·044 (107)	+·098±·065 (107)	—
	Total	−·224±·047 (187)		+·606±·031 (187)	+·092±·049 (187)	+·310±·054 (127)
M_2 ∠	Normal	−·208±·093 (48)	+·632±·058 (48)	—	+·533±·070 (48)	—
	Mesio-oblique impaction	−·184±·063 (107)	+·569±·044 (107)		−·112±·064 (107)	—
	Total	−·073±·049 (187)	+·606±·031 (187)		−·103±·049 (187)	−·054±·060 (127)
M_3 ∠	Normal	−·350±·085 (48)	+·343±·086 (48)	+·533±·070 (48)	—	—
	Mesio-oblique impaction	−·311±·059 (107)	+·098±·065 (107)	−·112±·064 (107)		—
	Total	−·437±·040 (187)	+·092±·049 (187)	−·103±·049 (187)		−·007±·060 (127)
Alv. margin ∠	Total	−·203±·057 (127)	+·310±·054 (127)	−·054±·060 (127)	−·007±·060 (127)	—

expected, the correlations for the normal and mesio-oblique impaction groups are close—no significant differences being found—except when $M_3\angle$ is one of the characters correlated. All the significant r's in the table are probably too low numerically, and by appreciable amounts, owing to "errors" of measurement. The highest values are between $M_1\angle$ and $M_2\angle$: there is a very close relation between the inclinations of the first and second molars, the first being normally more vertical with regard to the base-line than the second (see Table VII), and this does not seem to be disturbed by the eruption of the third molar. The two front teeth appear, indeed, to be little disturbed by the eruption of the last one. In cases of normal eruption the ultimate inclination of the third molar is quite intimately related to those of the first and second, but this association is destroyed entirely when there is abnormal eruption leading to ultimate impaction. The third molar space index is seen to be practically uncorrelated with $M_1\angle$ and $M_2\angle$, and there is no reason to believe that impaction is influenced at all by the inclinations of the front molars before the third erupts. The third molar space index and $M_3\angle$ are correlated significantly, however, and it may be asked whether any advantage would be gained by considering $M_3\angle$ in conjunction with the index in dealing with the problem of prediction. This possibility was explored by constructing the bi-variate (scatter) diagrams for the two characters in the case of each age-group considered separately, but there was no suggestion from these that a better separation between the classes of the condition of the third molar could be obtained from the two characters than from the index considered alone. It must be concluded that the angular measurements are not capable of aiding the purpose in view.

A similar treatment of the two indices suggests that there is no evidence to show that max. B, M_3/max. B, M_1 changes with age (for the range of ages represented), or that it is different for groups of mandibles exhibiting different conditions of the third molar. The corpus height index is obtained by dividing a length between the anterior outline of the first molar and the posterior outline of the second molar by a measurement of the height of the corpus. As soon as the second molar is completely erupted—i.e. at an age earlier than any considered here—the length in question may be supposed fixed, so any increase in the height due to growth will lead to a decrease in the index. It is actually found that the mean for the age-group 19, 18 and 17 is significantly greater than those for the age-groups 21 and 20, and 22 and over. Also, the mean for the total group of cases of mesio-oblique impaction (this having the larger proportion of younger individuals) is significantly greater than that for the total normal group. Constants for the corpus height index in Table IX relate to tracings for individuals aged 20 and over only. It is somewhat surprising to find that the mean for the other index considered there is very significantly greater than 100, signifying that usually the third molar is broader than the first*. The only suggestive correlation in the table is that between the index max. B, M_3/max. B, M_1 and the third molar space index. This is not significant as it stands, but it is quite likely that it would be found to

* No cases of peculiarly small molars sometimes found occurred in the series.

be higher if "errors" of measurement could be reduced. The positive sign indicates that more severe impaction (as judged by the third molar space index) is associated with a greater breadth of the third molar relative to the first. The correlation is spurious, however, since the breadth of the third molar is the numerator of both indices.

(8) *Summary and Conclusions.* The anatomical and clinical aspects and sequelae of the incomplete eruption of the lower wisdom tooth—a condition known as impaction—are discussed in the introduction. It is known that the sizes of the teeth are determined in the individual long before growth of the mandible has ceased, and the primary cause of impaction must be considered to be a failure of the bone to grow sufficiently to allow the last tooth to erupt completely and adopt the position which is normal for an adult. Instead it may be permanently retained in a position

TABLE IX.

Constants for and Correlations with two Indices.

	Means	Standard deviations	Correlations			
			$100 \times \frac{\text{Max. } B, M_3}{\text{Max. } B, M_1}$	Third molar space index	M_1 ∠	M_2 ∠
$100 \times \frac{\text{Max. } B, M_3}{\text{Max. } B, M_1}$ (all ages)	108·0 ± ·47 (187)	9·57 ± ·33	—	+ ·224 ± ·081 (62)†	—	—
Corpus height index (ages 20 and over)	72·0 ± ·69* (65)	8·24 ± ·49*	− ·030 ± ·084 (65)	+ ·167 ± ·081 (65)	+ ·142 ± ·082 (65)	− ·082 ± ·083 (65)

* For the 109 tracings relating to individuals of all ages the mean is 75·4 ± ·71 and the standard deviation 10·93 ± ·50.
† Only for individuals aged 22 and over.

assumed at one stage of normal development, or be deflected into a grossly abnormal position, as when it becomes horizontal instead of vertical. The impacted tooth may be partially erupted through the gum or it may remain completely buried, and the region is liable to become a focus of infection which may spread and lead to serious, and even fatal, complications. Removal of the impacted tooth leads to recovery, but this operation is often difficult and not without danger when infection has supervened. If a reliable method of predicting, in the case of an immature individual, that one or both of his lower third molars would ultimately become impacted, then the tooth could be removed at once as a preventive measure. The object of the investigation described in this paper was to explore the possibility of discovering such a method by obtaining measurements, in an indirect way, from radiographs of the lower molar regions of the jaws of a number of people of different ages.

The method used was to trace the outlines of the three molars and their pulp-cavities and certain margins of the mandible directly from the films, to enlarge these tracings with the aid of an enlarging camera, to obtain certain measurements from the enlarged tracings and to treat these measurements by statistical means. Only measurements of shape (indices and angles) can be used as the scales of enlargement are different for different individuals and unknown. The films used were all taken by the same extra-oral method, illustrated in Plate III C, and the main series consists of 187 relating to 127 individuals between the ages 17 and 42. The lines traced from the films and the methods of obtaining measurements from the enlarged tracings are described in sections 3 and 4 and illustrated by Figs. 2—6, which reproduce tracings actually used. Classes of impaction and other conditions of the third molar are defined, for the purpose of this enquiry, by considering geometrical relationships between certain lines traced, or afterwards drawn, on the tracings (section 5). Two sets of duplicated measurements were obtained—one from the same films and the other from repeated films for the same individuals—and these give crude estimates of the accuracy of the readings. Six characters relating to the outlines of the teeth were discarded entirely, as "errors" made in determining them were too large to make any statistical treatment profitable. Three indices and four angles were retained, although the errors for some of these are large enough to disturb statistical comparisons very appreciably in the case of the present material. A more stringent selection of the films used would ensure greater accuracy. One index—called the third molar space index—is discussed in detail. This expresses the maximum breadth of the outline of the third molar as a percentage of the space between the second molar and the border of the ramus, and errors made in determining it are of little importance. It decreases with age and makes a clear distinction between cases of normal eruption and impaction for individuals aged 22 and over. It is suggested that it may be accepted as a measure of the severity of impaction. If this view is correct, a solution of the problem of predicting ultimate impaction might be obtained from a knowledge of the way in which the value of the index changes with age. This could only be ascertained satisfactorily from records for the same individuals at different ages, and those available are not of this kind. By using an indirect method, the nature of the normal age change in the index is inferred, but no appreciation can be obtained from the material used of individual differences in the growth process. There is no suggestion that any of the other measurements will be found of use in aiding a solution of the problem if considered in conjunction with the third molar space index, but they are of considerable interest apart from this question, and it is suggested that it would be profitable to determine these measurements for more ample material. This only purports to be a preliminary investigation, and the results seem to justify the conclusion that suitable material worked up on the same lines would lead to a satisfactory solution of the problem discussed. The need is for repeated radiographs of a sample of 100 individuals, say, taken at intervals of a year between their 16th and 22nd birthdays. More suitable films than some of those used in this paper would be required, and the metrical results obtained should be correlated with clinical records.

DESCRIPTION OF PLATES.

Plate I. *Cases of Paroxysmal Neuralgia.*

A. Skiagram of the jaws of a man, aged 30, who had suffered for many years from severe paroxysmal neuralgia on the left side of the face and neck, and also from spasmodic trismus preventing mastication. His teeth, generally, were in a good state of preservation and unusually free from decay. Radiographic examination showed an impacted wisdom tooth impinging against the adjacent molar and giving rise to carious absorption of the crown. The carious area had spread gradually through the sensient dentine to the pulp of the tooth, from the irritation of the nerves of which odontalgia and reflex spasm of the masticatory muscles originated. This plate also shows the proximity of the mandibular nerve to the root of the tooth, which in some cases is grooved, or even tunnelled, for its accommodation.

B. Skiagram of the mandible of a man, aged 65, who was apparently edentulous, but suffered from severe neuralgia and attacks of giddiness. These symptoms were due to infection in the socket of an embedded third molar which was irritating the main trunk of the mandibular nerve, a branch of the third division of the trigeminus. The appearance of white cloudiness in the skiagram is due to sclerosing osteitis of the mandible, and it affords evidence of long-standing chronic infection. In a mandible of this nature extraction of the tooth is difficult, the bone is more brittle and hence more liable to fracture, and there is diminished resistance to post-operative infection leading to much after-pain and even to pathological fracture. Healing of the socket is slow. A photograph of the extracted tooth (Plate I C) shows a deep groove formed during calcification of the tooth. The nerve lay in this recess.

C. An enlarged photograph showing the posterior view of the extracted third molar seen laterally *in situ* in Plate I B. The mandibular nerve passed through the groove formed in the root.

Plate II. *Drawings of Types of Impaction of the Lower Third Molar.*

A. A drawing of the mouth showing partial emergence of the lower wisdom tooth. The posterior portion of the crown is covered by the mucosal *capuchon*, which has become inflamed and swollen as a result of infection in the confined space between the gum and the covered portion of the tooth and has produced an abscess of the follicle.

B. A sectional drawing of the molar region illustrating the same condition and showing the focus of infection.

C. [Reproduced with the kind permission of the Editor of *The Medical Press.*] A diagrammatic scheme of the main types of impaction of the mandibular third molar, showing the relation of this tooth to the second molar, to the coronoid process, and to the mandibular canal. The tooth is not always situated directly behind the second molar, but as well as being impacted and rotated in an antero-posterior direction it may also be deflected towards the tongue or towards the cheek.

Plate III. *Natives of Ceylon suffering severely from Third Molar Infection, and an illustration of the Method of taking a Radiograph of the Right Side of the Jaws.*

A. Photograph of a patient suffering from acute infection of the follicle of an impacted lower third molar. A condition of osteomyelitis with cellulitis in the soft tissues has necessitated external drainage.

B. Photograph of a patient suffering from acute infection of the follicle of an impacted lower third molar which has caused thrombophlebitis of the cavernous sinus: death ensued. The writers are indebted to Dr W. Balendra (Ceylon) for permission to use the photographs reproduced in Plate III A and B.

C. Photograph illustrating the method of taking a radiograph of the right side of the jaws (see pp. 390—391 of text). This is reproduced with the kind permission of Dr S. Blackman from his paper cited above (footnote to p. 390).

Plate IV. *Traumatic Fractures of the Mandible in the Third Molar Region.*

A. Skiagram of the mandible of an edentulous man, aged 53, showing a fracture of the bone in the third molar region due to a blow. This is a position of inherent weakness.

B. Skiagram of the mandible of a man, aged 40, who received a blow on the side of the face. The mandible was fractured through the socket of the wisdom tooth which was also split.

C. Skiagram of the mandible of a woman, aged 41, apparently edentulous. In a motor accident she received a blow on the chin and fracture of the mandible occurred through the part which was weakened owing to the retention of a buried third molar.

Plate V. *Traumatic Fractures of the Mandible through the Third Molar Crypt.*

A. Skiagram taken antero-posteriorly of the jaws of a man, aged 20, showing a traumatic fracture of the left side of the mandible passing through the crypt of the developing third molar.

B. Skiagram of the mandible of a soldier, aged 20, showing a fracture, received during boxing, which passes through the crypt of the impacted and semi-formed third molar.

Plate VI. *Operative Fracture of a Mandible during Extraction of a Third Molar.*

A. Skiagram of the mandible of a woman, aged 48. Fracture of the bone occurred during the removal of the buried third molar.

B. Skiagram of the same patient made 35 days after the fracture of the mandible.

Plate VII. *Pathological Fracture following Extraction of a Third Molar.*

A. Skiagram of part of the mandible of a man, aged 53, showing the buried third molar *in situ*.

B. Skiagram of the region of the socket of the tooth after extraction, showing that the mandible was not fractured during the operation.

C. An antero-posterior skiagram of the same case eight months later and during treatment. Sub-acute osteomyelitis had supervened after the extraction, and necrosis with spontaneous fracture of the mandible occurred eight weeks after the operation. The skiagram shows the major portion of the mandible fixed to the maxilla by an interdental metal splint. The gap due to separation of the fragments of the lower jaw by muscle-pull and loss of bone by necrosis at the site of the fracture is seen on the left.

Plate VIII. *Pathological and Anomalous Jaws.*

A. Skiagram of the jaws (sex and age of individual unknown) showing extensive destruction of the bone of the mandible following cystic distension of the buried third molar follicle.

B. Skiagram of the jaws of a man, aged 21, showing reduction in size of the maxillary third molar and impaction of the mandibular third molar which is of normal size.

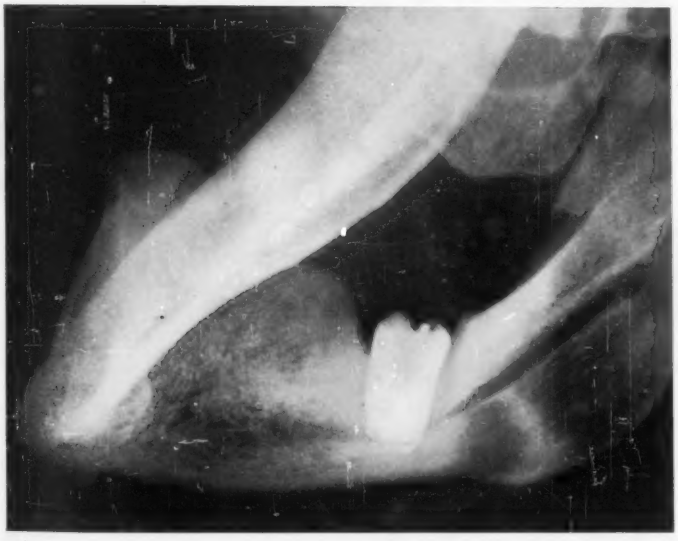
Plate IX. *Four Stages of Dental Development.*

These skiagrams of four skulls are reproduced, with permission, from *An Atlas of Skiagrams illustrating the Development of the Teeth*, by Johnson Symington and J. C. Rankin, 1908. The ages of the specimens are: A, 3 years; B, 10 years; C, 13 years and D, adult.

Plate X. *Normal Rotation and Eruption of the Lower Third Molars in Rhesus Macaque.*

[Reproduced with the kind permission of the Editor of *The Dental Record*.] Bilateral skiagrams of the molar regions of the mandibles of three Rhesus Macaque monkeys, showing normal development and eruption of M_2 and M_3. In the uppermost pair of skiagrams M_2, its roots not yet formed, is mesially tilted against the distal surface of M_1: M_3 is seen posteriorly, the cusps of the occlusal surface only being calcified. In the middle skiagrams M_2 has erupted normally and is in full occlusion: M_3 is two-thirds formed, and is in a position of transitory mesio-oblique impaction with M_2. There is, however, an ample sufficiency of room and a space exists between M_2 and M_3. In the lowermost pair of skiagrams M_3 has erupted normally, and is in full occlusion. In the cases of both M_2 and M_3 normal rotation of the molars from mesio-oblique inclination to the vertical attitude has taken place unhindered by limitation of growth.

A. Caries of M_2 due to Impacted M_3.



B. Unerupted M_3 embracing the Mandibular Nerve.

C. The extracted M_3 (posterior view).

Cases of Paroxysmal Neuralgia.

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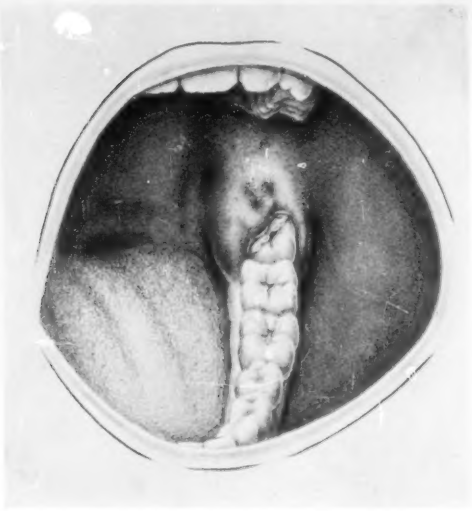
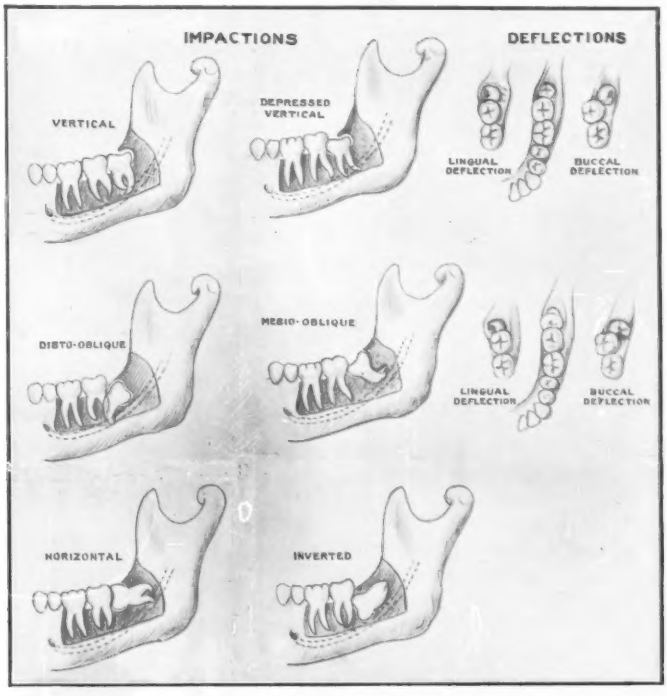


A. Occlusal View of Soft Tissue Impaction with Follicular Abscess.

B. Section of Molar Region illustrating the same Condition.



C. Various Types of Impaction.

Drawings of Types of Impaction of the Lower Third Molar.

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A. Osteomyelitis and Cellulitis due to Follicular Infection of M_3.

B. Infection of M_3 leading to Thrombophlebitis of Cavernous Sinus and Death.

C. Illustrating Method of taking a Radiograph of the Right Side of the Jaws.

Natives of Ceylon suffering severely from Third Molar Infection, and an Illustration of the Method of taking a Radiograph of the Right Side of the Jaws.

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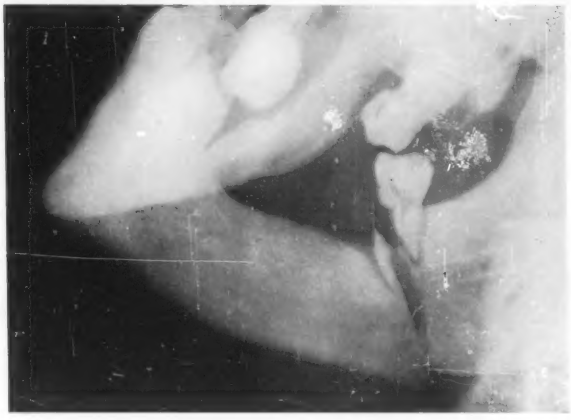
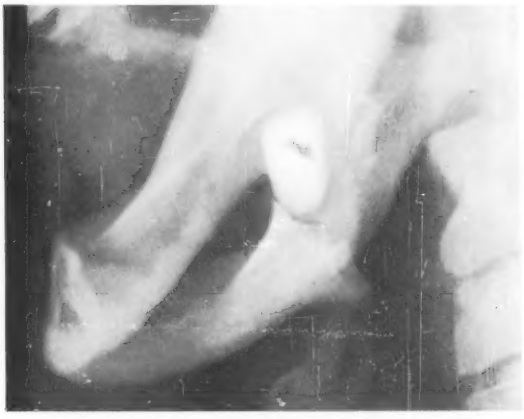
A. A Man aged 53.

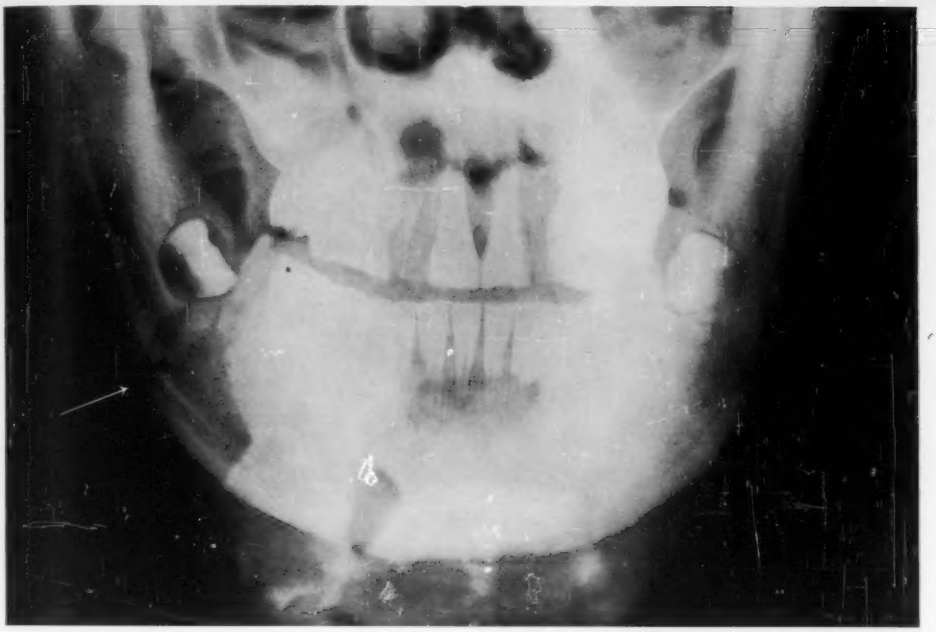


B. A Man aged 40: showing split M_3.

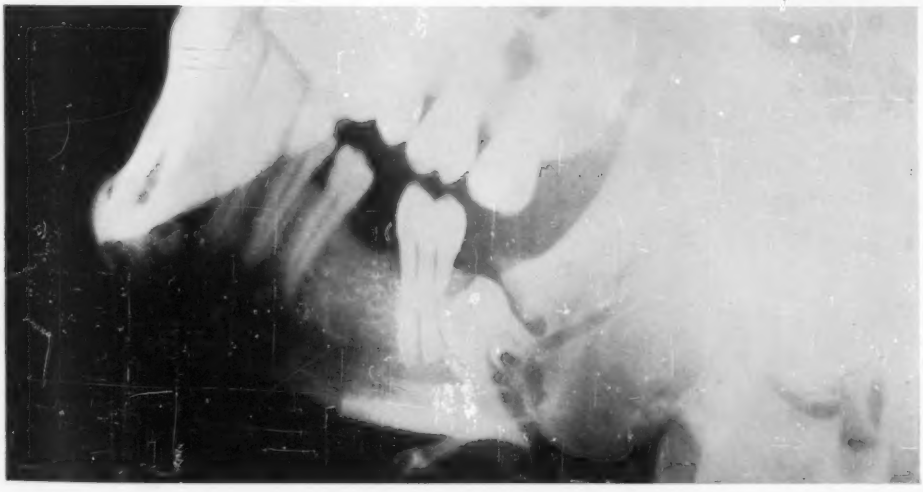


C. A Woman aged 41.

Traumatic Fractures of the Mandible in the Third Molar Region.



A. A Man aged 20.



B. A Man aged 20.

Traumatic Fractures of the Mandible through the Third Molar Crypt.

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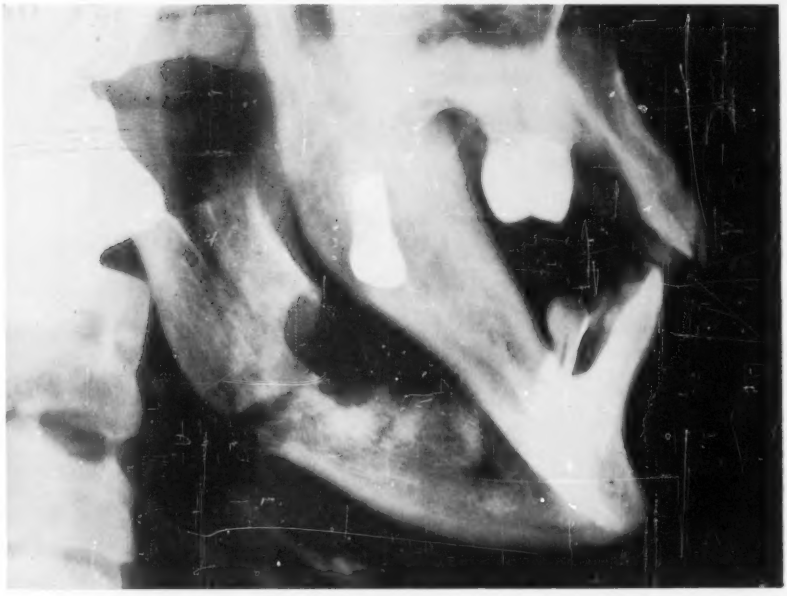
A. Before Extraction.



B. Fractured Mandible after Extraction.

Operative Fracture of a Mandible during Extraction of a Third Molar.

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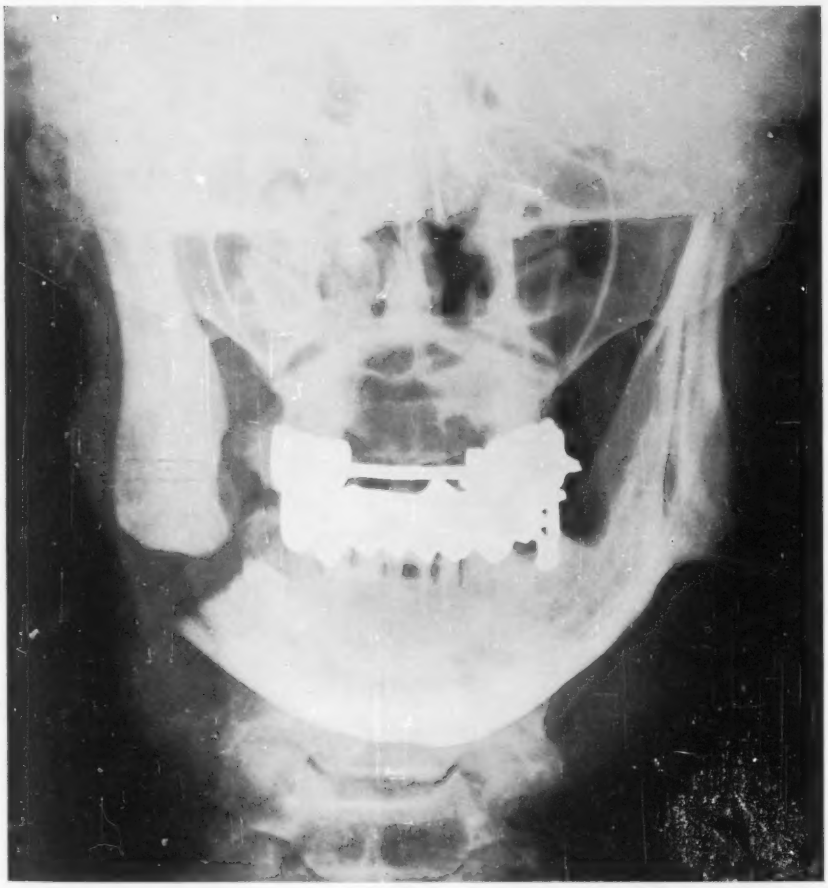
A. The Third Molar *in situ*.

B. The Socket of the Tooth after extraction.



C. Antero-Posterior View showing Fracture and Splint.

Pathological Fracture following Extraction of a Third Molar.

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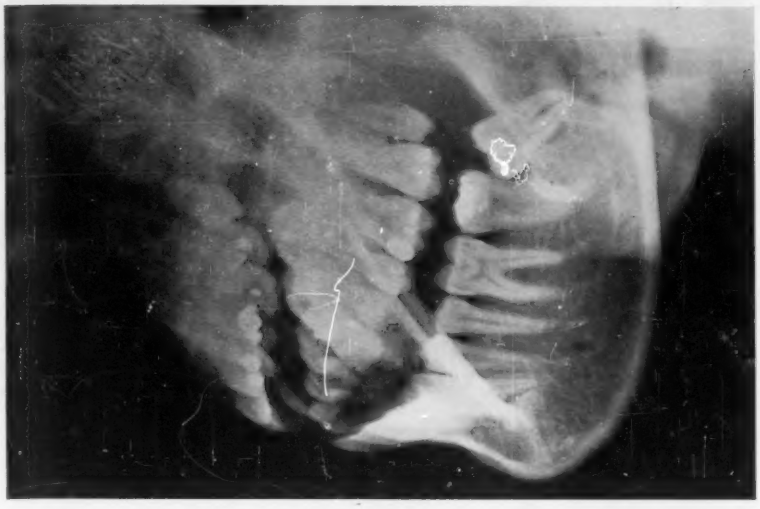
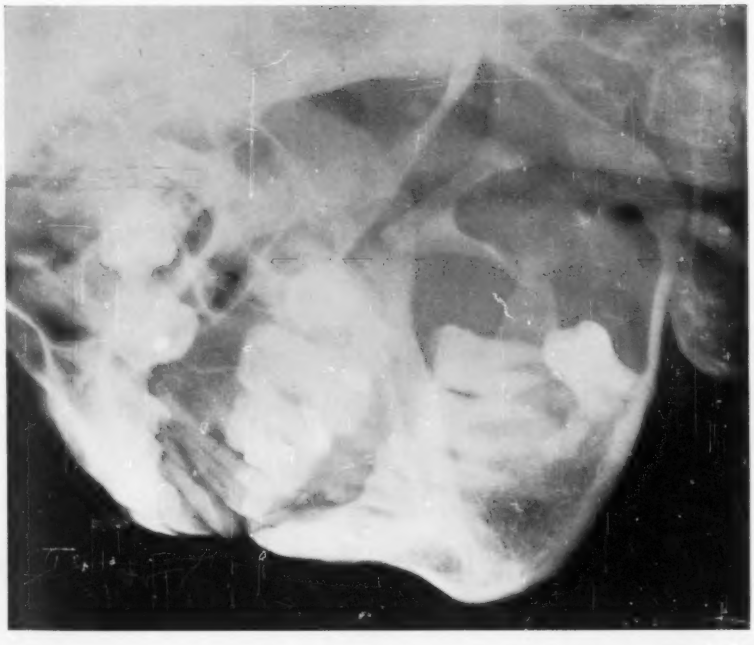


B. Impaction of M_3 associated with Reduced M^3.



A. Dentigerous Cyst of M_3.

Pathological and Anomalous Jaws.

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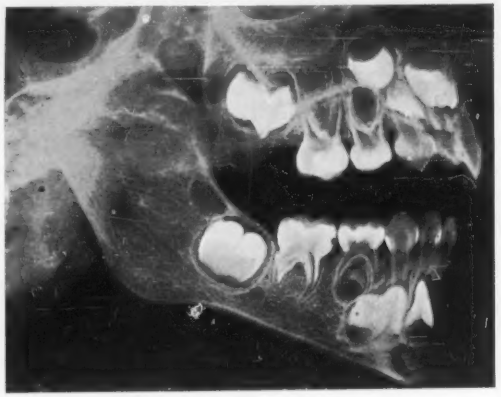

A. Age 3.

B. Age 10.

C. Age 13.

D. Adult.

Four Stages of Dental Development (after Symington and Rankin).

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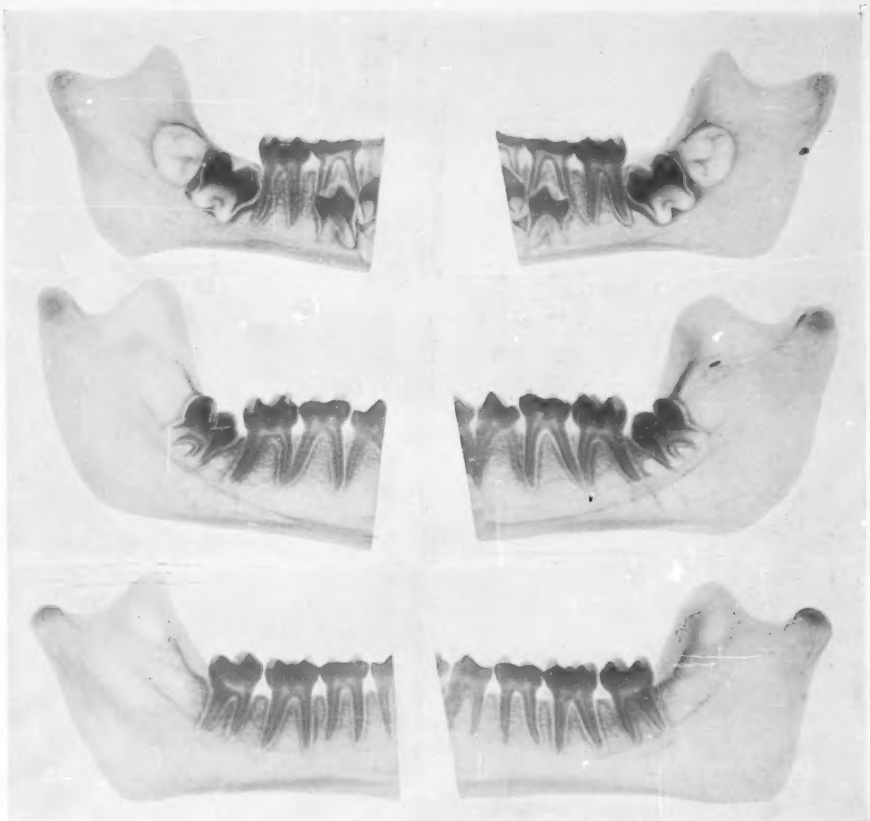

Normal Rotation and Eruption of the Lower Third Molars in Rhesus Macaque.

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A. Bi-molar Skiagram showing: Right, forward Position of M_3 due to early Extraction of M_2; Left, Impacted M_3.

B. Extreme Tilting of the Crown of M_3 in a Child of $9\frac{1}{2}$ years.

C. The same Subject 22 months after Enucleation of the partly-formed M_3.

Illustrations of Unusual Conditions of the Third Molar Region.

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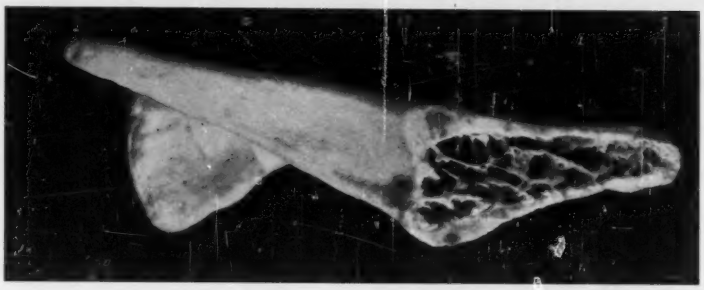
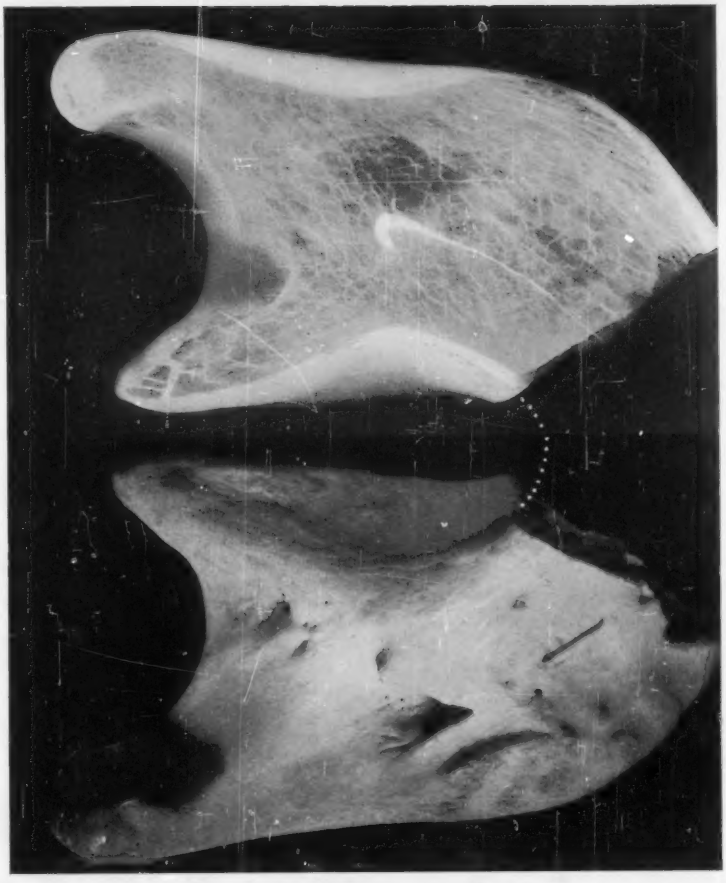


B. Anterior Aspect of the same Fragment.



A. Juxtaposed Photograph and Skiagram.

Photographs and a Skiagram of the Left Ramus of a Mandible.

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Bio

A. Male aged 20; M_3 erupted normally.

B. Male aged 26; M_3 erupted normally.

C. Male aged 18; mesio-oblique impaction.

D. Male aged 22; excessive eruption of M_3.

Illustrations of different Conditions of the Mandibular Third Molar.

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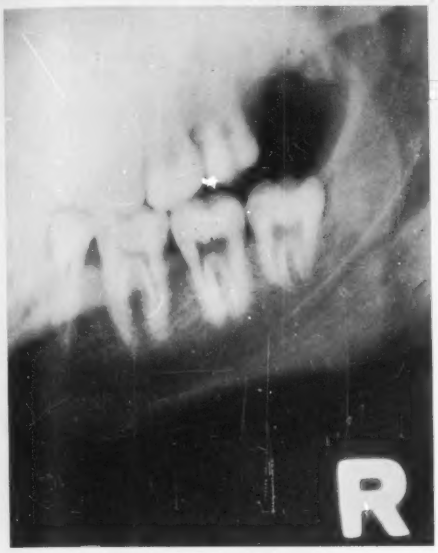

A. Male aged 22; coronoid impaction.

B. Male aged 21; horizontal impaction.

C. Female aged 22; mesio-oblique impaction. Showing vertebrae obscuring molar region and artificial filling of teeth.

D. Male aged 24; coronoid or depressed vertical impaction. Showing other side of mandible obscuring molar region and teeth clenched.

Illustrations of different Conditions of the Mandibular Third Molar and other Points.

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Plate XI. *Illustrations of Unusual Conditions of the Third Molar Region.*

A. The bi-molar skiagram of a male student, aged 22 years 8 months, whose right mandibular first molar was extracted on account of caries at the age of 12 years. The second molar has swung forward and room has thus been gained posteriorly to it for the full eruption of the wisdom tooth. On the left side all the premolars and molars are present and the third molar is impacted against the second molar.

B and C. [Reproduced with the kind permission of the Editor of *The Lancet*.] Two skiagrams of the same patient before and after prophylactic enucleation of the right mandibular third molar in an early stage of development. B is a skiagram of a child of 9½ years showing extreme mesial tilting of the crown of M_3, of which the occlusal surface only is calcified. It was considered that, as the chance of full eruption of this tooth was remote, it would be preferable to enucleate it. C is a skiagram of the same case made 22 months after operation. The upper and lower second molars are seen to have erupted into normal occlusion in the interim.

Plate XII. *Photographs and a Skiagram of the Left Ramus of a Mandible.*

A. Juxtaposed enlarged lateral photograph and skiagram showing the correspondence of the rameal line traced and other markings (see p. 395 of text).

B. An antero-posterior photograph of the same fragment.

Plate XIII. *Illustrations of different Conditions of the Mandibular Third Molar.*

These are taken from the series of skiagrams from which measurements were obtained, and the enlarged tracings corresponding to them are Figs. 2—5 in the text.

A. Male aged 20; M_3 erupted normally (Fig. 2).

B. Male aged 26; M_3 erupted normally (Fig. 3).

C. Male aged 18; mesio-oblique impaction (Fig. 4).

D. Male aged 22; excessive eruption of M_3 (Fig. 5).

Plate XIV. *Illustrations of different Conditions of the Mandibular Third Molar and other Points.*

These are taken from the series of skiagrams used: measurements were obtained from A and B and the enlarged tracing for A is Fig. 6 in the text; no measurements were obtained from C and D for the reasons stated below.

A. Male aged 22; coronoid impaction (Fig. 6).

B. Male aged 21; horizontal impaction, an extreme form of mesio-oblique impaction. Skiagram reversed.

C. Female aged 22; mesio-oblique impaction. No reliable measurements can be obtained from this film, since images of the cervical vertebrae obscure the outlines of the third molar region of the mandible, and also because the superior outlines of the lower first and second molars are uncertain owing to artificial filling of the teeth.

D. Male aged 24; coronoid or depressed vertical impaction. No reliable measurements can be obtained from this film since the image of the lower first molar is obscured by that of the other side of the mandible and also because the teeth are clenched.

THE STANDARD ERROR OF GINI'S MEAN DIFFERENCE.

By U. S. NAIR.

1. *Introductory.*

The Mean Difference has been introduced by Professor Corrado Gini as a measure of variation*. It is defined as the mean value of all the possible differences between individual observations. Thus, if $x_1, x_2, \ldots x_n$ are n observed values of a variate x, the mean difference is defined as

$$g = \frac{2}{n(n-1)} \sum_{i \neq j} |x_i - x_j|.$$

In the present paper, expressions will be derived for the standard error of g when the variate follows any continuous distribution law, and, in particular, explicit formulae will be given for the normal, the exponential and the rectangular distributions. In these three cases, it is possible to use the value of g calculated from a sample to estimate the value of the parameter measuring the scale of variation in the population. The reliability of this form of estimate will be compared with that of other alternatives.

2. *The standard error of the mean difference.*

If $x_1, x_2, \ldots x_n$, are n observed values of a variate x, arranged in ascending order of magnitude, the mean difference g, may be written as follows:

$$\begin{aligned} g &= \frac{2}{n(n-1)} \sum_{i=1}^{n} [(x_i - x_1) + (x_i - x_2) + \ldots + (x_i - x_{i-1})] \\ &= \frac{2}{n(n-1)} \left[2 \sum_{i=1}^{n} i x_i - (n+1) \sum_{i=1}^{n} x_i \right] \\ &= \frac{2}{n(n-1)} [2U - (n+1)V] \quad \ldots\ldots(1), \end{aligned}$$

where $$U = \sum_{i=1}^{n} i x_i \quad \ldots\ldots(2),$$

and $$V = \sum_{i=1}^{n} x_i \quad \ldots\ldots(3).$$

To calculate the mean value and the standard deviation of g in repeated samples, it is sufficient to find the expectations of U, V, U^2, UV and V^2. These may be evaluated as follows:

* Corrado Gini, "Sulla misura della concentrazione e della variabilità dei caratteri." *Atti del R. Instituto Ven. di S.L.A.* (1913—14), Vol. 73, II. For full references see C. Gini, "Intorno alle curve di concentrazione," *Metron.*, Vol. IX., no. 3—4 (1932), p. 3.

Suppose that the variable, x, must lie between a and b and that $p(x)$ is its probability law so that

$$\int_a^b p(x)\,dx = 1.$$

Then the probability that the ith observation in order of magnitude in a series of n observations has a value x, $(a \leqslant x \leqslant b)$, is

$$\frac{n!}{(i-1)!\,(n-i)!}\,p(x)\,A_x^{\,i-1}(1-A_x)^{n-i} \quad \ldots\ldots(4),$$

where

$$A_x = \int_a^x p(x)\,dx \quad \ldots\ldots(5).$$

(i) Hence if $\overline{U}$ is the expectation of U,

$$\begin{aligned}\overline{U} &= \int_a^b \left[\sum_{i=1}^{n} ix\,\frac{n!}{(i-1)!\,(n-i)!}\,A_x^{\,i-1}(1-A_x)^{n-i}p(x)\right]dx \\ &= n\int_a^b xp(x)\left[\sum_{i=1}^{n}\frac{(n-1)!\,i}{(i-1)!\,(n-i)!}\,A_x^{\,i-1}(1-A_x)^{n-i}\right]dx \\ &= n\int_a^b xp(x)\,[1+(n-1)\,A_x]\,dx \quad \ldots\ldots(6)^*.\end{aligned}$$

If $\overline{V}$ is the expectation of V, it is easily seen that

$$\overline{V} = n\int_a^b xp(x)\,dx \quad \ldots\ldots(7).$$

From (1) $\bar{g}$, the expectation of g, is given by

$$\bar{g} = \frac{2}{n(n-1)}\,[2\overline{U} - (n+1)\,\overline{V}] \quad \ldots\ldots(8),$$

and (6), (7) and (8) give

$$\bar{g} = 2\int_a^b xp(x)\,[2A_x - 1]\,dx \quad \ldots\ldots(9).$$

(ii) Squaring both sides of (2),

$$\begin{aligned}U^2 &= \sum_{i=1}^{n} i^2x_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} ijx_ix_j \\ &= S_1 + 2S_2 \quad \ldots\ldots(10),\end{aligned}$$

where

$$S_1 = \sum_{i=1}^{n} i^2x_i^2 \quad \ldots\ldots(11),$$

and

$$S_2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} ijx_ix_j \quad \ldots\ldots(12).$$

Hence if $\overline{U^2}$, $\overline{S_1}$ and $\overline{S_2}$ denote the expectations of U^2, S_1 and S_2 respectively,

$$\overline{U^2} = \overline{S_1} + 2\overline{S_2} \quad \ldots\ldots(13).$$

* This result follows because

$$\sum_{i=1}^{n}\frac{(n-1)!}{(i-1)!\,(n-i)!}\,iA_x^{\,i-1}(1-A_x)^{n-1} = \frac{d}{d\theta}\left[\theta\,(\theta+\phi)^{n-1}\right]\Big|_{\substack{\theta=A_x\\ \phi=1-A_x}}.$$

Now,
$$\overline{S_1} = \int_a^b \left[\sum_{i=1}^{n} i^2 x^2 \frac{n!}{(i-1)!\,(n-i)!} A_x^{i-1} (1-A_x)^{n-i} p(x) \right] dx$$
$$= n \int_a^b x^2 p(x) \left[\sum_{i=1}^{n} \frac{(n-1)!\, i^2}{(i-1)!\,(n-i)!} A_x^{i-1} (1-A_x)^{n-i} \right] dx$$
$$= n \int_a^b x^2 p(x) [1 + 3(n-1) A_x + (n-1)(n-2) A_x^2]\, dx \quad \ldots\ldots(14)^*.$$

To find $\overline{S_2}$, we make use of the fact that the probability that the ith and jth observations in order of magnitude in a series of n observations have values x_1 and x_2 respectively, is

$$\frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} A_{x_1}^{i-1} (A_{x_2} - A_{x_1})^{j-i-1} (1 - A_{x_2})^{n-j} p(x_1)\, p(x_2) \quad (15),$$

A_{x_1} and A_{x_2} being the values of A_x in (5) when $x = x_1$ and x_2 respectively. Therefore,

$$\overline{S_2} = \int_a^b \left[\int_a^{x_2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} ij x_1 x_2 \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \times A_{x_1}^{i-1} (A_{x_2} - A_{x_1})^{j-i-1} (1 - A_{x_2})^{n-j} p(x_1)\, p(x_2)\, dx_1 \right] dx_2$$
$$= n \int_a^b x_2 p(x_2) \left[\int_a^{x_2} x_1 p(x_1) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(n-1)!\, ij}{(i-1)!\,(j-i-1)!\,(n-j)!} \times A_{x_1}^{i-1} (A_{x_2} - A_{x_1})^{j-i-1} (1 - A_{x_2})^{n-j}\, dx_1 \right] dx_2$$
$$= n \int_a^b x_2 p(x_2) \left[\int_a^{x_2} x_1 p(x_1)\, T dx_1 \right] dx_2 \quad \ldots\ldots(16),$$

where
$$T = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(n-1)!\, ij}{(i-1)!\,(j-i-1)!\,(n-j)!} A_{x_1}^{i-1} (A_{x_2} - A_{x_1})^{j-i-1} (1 - A_{x_2})^{n-j} \quad \ldots\ldots(17).$$

By methods similar to those employed to obtain (6) and (14), the sum on the right side of (17) may be evaluated and we find that

$$T = (n-1)[2 + 3(n-2) A_{x_1} + (n-2) A_{x_2} + (n-2)(n-3) A_{x_1} A_{x_2}].$$

Substituting the value of T in (16),

$$\overline{S_2} = n(n-1) \int_a^b x_2 p(x_2) \left[\int_a^{x_2} x_1 p(x_1) \left\{ 2 + 3(n-2) A_{x_1} + (n-2) A_{x_2} + (n-2)(n-3) A_{x_1} A_{x_2} \right\} dx_1 \right] dx_2 \quad \ldots\ldots(18).$$

* This follows because
$$\sum_{i=1}^{n} \frac{(n-1)!\, i^2}{(i-1)!\,(n-i)!} A_x^{i-1} (1-A_x)^{n-1} = \frac{d}{d\theta} \left[\theta \frac{d}{d\theta} \right] [\theta (\theta + \phi)^{n-1}] \Bigg|_{\substack{\theta = A_x \\ \phi = 1 - A_x}}.$$

From (10), (14) and (18), we find that

$$\overline{U^2} = n\int_a^b x^2 p(x)\,[1+3(n-1)A_x+(n-1)(n-2)A_x^2]\,dx$$
$$+ 2n(n-1)\int_a^b x_2 p(x_2)\left[\int_a^{x_2} x_1 p(x_1)\left\{2+3(n-2)A_{x_1}+(n-2)A_{x_2}\right.\right.$$
$$\left.\left.+(n-2)(n-3)A_{x_1}A_{x_2}\right\}dx_1\right]dx_2 \quad \ldots\ldots(19).$$

(iii) Multiplying (2) and (3),

$$UV = \sum_{i=1}^{n} ix_i^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i+j)x_ix_j$$
$$= S_3 + S_4 \quad \ldots\ldots(20),$$

where
$$S_3 = \sum_{i=1}^{n} ix_i^2 \quad \ldots\ldots(21),$$

and
$$S_4 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i+j)x_ix_j \quad \ldots\ldots(22).$$

Hence the expectation of UV is given by

$$\overline{UV} = \overline{S_3} + \overline{S_4} \quad \ldots\ldots(23),$$

$$\overline{S_3} = \int_a^b \left[\sum_{i=1}^{n} ix^2 \frac{n!}{(i-1)!\,(n-i)!} A_x^{i-1}(1-A_x)^{n-i}p(x)\right]dx$$
$$= n\int_a^b x^2 p(x)\left[\sum_{i=1}^{n}\frac{(n-1)!\,i}{(i-1)!\,(n-i)!}A_x^{i-1}(1-A_x)^{n-i}\right]dx$$
$$= n\int_a^b x^2 p(x)\,[1+(n-1)A_x]\,dx \quad \ldots\ldots(24).$$

Similarly,

$$\overline{S_4} = \int_a^b \left[\int_a^{x_2} x_1x_2 p(x_1)\,p(x_2)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{n!\,(i+j)}{(i-1)!\,(j-i-1)!\,(n-j)!}\right.$$
$$\left.\times A_{x_1}^{\,i-1}(A_{x_2}-A_{x_1})^{j-i-1}(1-A_{x_2})^{n-j}\,dx_1\right]dx_2$$
$$= n(n-1)\int_a^b x_2 p(x_2)\left[\int_a^{x_2} x_1 p(x_1)\,\{3+(n-2)(A_{x_1}+A_{x_2})\}\,dx_1\right]dx_2 \quad \ldots(25).$$

Substituting the values of $\overline{S_3}$ and $\overline{S_4}$ in (23),

$$\overline{UV} = n\int_a^b x^2 p(x)\,[1+(n-1)A_x]\,dx$$
$$+ n(n-1)\int_a^b x_2 p(x_2)\left[\int_a^{x_2} x_1 p(x_1)\,\{3+(n-2)(A_{x_1}+A_{x_2})\}\,dx_1\right]dx_2 \quad \ldots(26).$$

(iv) Squaring (3),

$$V^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_ix_j.$$

Proceeding in exactly the same way as for U^2, the expectation of V^2 is given by

$$\overline{V^2} = n\int_a^b x^2 p(x)\,dx + 2n(n-1)\int_a^b x_2 p(x_2)\left[\int_a^{x_2} x_1 p(x_1)\,dx_1\right]dx_2 \quad \ldots(27).$$

From (1), $$g^2 = \frac{4}{n^2(n-1)^2}[4U^2 - 4(n+1)UV + (n+1)^2V^2] \quad \ldots\ldots\ldots\ldots(28).$$

Therefore, $\overline{g^2}$, the expectation of g^2, is given by

$$\overline{g^2} = \frac{4}{n^2(n-1)^2}[4\overline{U^2} - 4(n+1)\overline{UV} + (n+1)^2\overline{V^2}].$$

Using the results in (19), (26) and (27),

$$\overline{g^2} = \frac{4}{n(n-1)}(I_1 + 2(n-2)I_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(29),$$

where

$$\left.\begin{aligned} I_1 &= \int_a^b x^2 p(x)[(n-1) - 4(n-2)A_x + 4(n-2)A_x^2]\,dx \\ \text{and} \quad I_2 &= \int_a^b x_2 p(x_2)\left[\int_a^{x_2} x_1 p(x_1)\{(n-3) - 2(n-5)A_{x_1} \right.\right. \\ &\qquad \left.\left. - 2(n-1)A_{x_2} + 4(n-3)A_{x_1}A_{x_2}\}\,dx_1\right]dx_2 \end{aligned}\right\} \ldots(30).$$

Denoting the standard error of g by σ_g, since

$$\sigma_g^2 = \overline{g^2} - \overline{g}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

σ_g^2 may be evaluated from (29)—(31) with the help of (9).

3. *Formulae for $\overline{g}$ and σ_g for the Normal, the Exponential and the Rectangular distributions.*

In this section, the values of $\overline{g}$ and σ_g are evaluated in the cases where

(i) $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ $(a = -\infty \leqslant x \leqslant \infty = b)$ $\ldots\ldots\ldots\ldots(32).$

(ii) $p(x) = \dfrac{1}{\sigma} e^{-x/\sigma}$ $(a = 0 \leqslant x \leqslant \infty = b)$ $\ldots\ldots\ldots\ldots\ldots(33).$

(iii) $p(x) = \dfrac{1}{k}$ $(a = 0 \leqslant x \leqslant k = b)$ $\ldots\ldots\ldots\ldots\ldots(34).$

The values of the integrals required are given in the following table. In the case of the exponential and the rectangular distributions, the integrals may be directly evaluated. For the normal curve, some transformations are needed. These are illustrated for the eighth integral in Table I. The method followed for the other integrals is similar*.

Let $$J = \int_a^b dx_2 \int_a^{x_2} x_1 x_2 A_{x_1} A_{x_2} p(x_1) p(x_2)\,dx_1$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} x_2 e^{-x_2^2/2\sigma^2} A_{x_2} dx_2 \int_{-\infty}^{x_2} x_1 e^{-x_1^2/2\sigma_2} A_{x_1} dx_1.$$

* For other integrals of this type reference may be made to a paper by T. Hojo, "Distribution of the Median, Quartiles and Inter-quartile Distance in Samples from a Normal Population." *Biometrika*, Vol. XXIII. (1931), p. 317.

TABLE I.

	Integral	Normal	Exponential	Rectangular
1	$\int_a^b x p(x) A_x dx$	$\frac{\sigma}{2\sqrt{2\pi}}$	$\frac{3\sigma}{4}$	$\frac{k}{3}$
2	$\int_a^b x^2 p(x)\, dx$	σ^2	$2\sigma^2$	$\frac{k^2}{3}$
3	$\int_a^b x^2 A_x p(x)\, dx$	$\frac{\sigma^2}{2}$	$\frac{7\sigma^2}{4}$	$\frac{k^2}{4}$
4	$\int_a^b x^2 A_x^2 p(x)\, dx$	$\frac{\sigma^2}{6\pi}(2\pi+\sqrt{3})$	$\frac{85\sigma^2}{54}$	$\frac{k^2}{5}$
5	$\int_a^b dx_2 \int_a^{x_2} x_1 x_2 p(x_1)\, p(x_2)\, dx_1 dx_2$	0	$\frac{\sigma^2}{2}$	$\frac{k^2}{8}$
6	$\int_a^b dx_2 \int_a^{x_2} x_1 x_2 A_{x_1} p(x_1)\, p(x_2)\, dx_1 dx_2$	$\frac{\sigma^2}{4\pi\sqrt{3}}$	$\frac{17\sigma^2}{54}$	$\frac{k^2}{15}$
7	$\int_a^b dx_2 \int_a^{x_2} x_1 x_2 A_{x_2} p(x_1)\, p(x_2)\, dx_1 dx_2$	$-\frac{\sigma^2}{4\pi\sqrt{3}}$	$\frac{47\sigma^2}{54}$	$\frac{k^2}{10}$
8	$\int_a^b dx_2 \int_a^{x_2} x_1 x_2 A_{x_1} A_{x_2} p(x_1)\, p(x_2)\, dx_1 dx_2$	$\frac{\sigma^2}{8\pi}$	$\frac{\sigma^2}{9}$	$\frac{k^2}{18}$

This integral may be transformed into

$$\frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} y A_y e^{-y^2/2}\, dy \int_{-\infty}^{y} x A_x e^{-x^2/2}\, dx,$$

where

$$A_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}\, dx.$$

Performing a partial integration on the second integral,

$$J = \frac{\sigma^2}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty} y e^{-y^2/2} A_y dy \int_{-\infty}^{y} e^{-x^2} dx - \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} y e^{-y^2} A_y{}^2 dy$$

$$= \frac{\sigma^2}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty} y e^{-y^2/2} A_y dy \int_{-\infty}^{y} e^{-x^2} dx - \frac{\sigma^2}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{-3y^2/2} A_y dy$$

$$= \frac{\sigma^2}{(2\pi)^{\frac{3}{2}}}[J_1 - J_2],$$

where

$$J_1 = \int_{-\infty}^{\infty} y e^{-y^2/2} A_y dy \int_{-\infty}^{y} e^{-x^2} dx \quad \text{and} \quad J_2 = \int_{-\infty}^{\infty} e^{-3y^2/2} A_y dy.$$

To evaluate J_1, replace x by $(y-x)$ and then replace y by $(y+x)$, so that

$$J_1 = \int_{-\infty}^{\infty} (y+x)\, e^{-(y+x)^2/2} A_{(y+x)}\, dy \int_0^{\infty} e^{-y^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-y^2} dy \int_0^{\infty} (y+x)\, A_{(y+x)}\, e^{-(y+x)^2/2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-y^2} dy \left[- A_{(y+x)}\, e^{-(y+x)^2/2} \Big|_{x=0}^{x=\infty} + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(y+x)^2} dx \right]$$

$$= \int_{-\infty}^{\infty} e^{-3y^2/2} A_y dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \int_0^{\infty} e^{-(y+x)^2} dx$$

$$= J_2 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2(y+(x/2))^2} dy \int_0^{\infty} e^{-x^2/2} dx.$$

Therefore $$J_1 - J_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \frac{dz}{2} \int_0^{\infty} e^{-x^2/2} dx = \tfrac{1}{4}\sqrt{2\pi}.$$

Thus $$J = \sigma^2/8\pi.$$

(i) *Normal distribution* [equation (32)].

Substituting the values of the integrals from the second column of Table I, into (9) and (29)—(31), we find

$$\bar{g} = \frac{2\sigma}{\sqrt{\pi}} \quad \text{.....(35),}$$

$$\sigma_g = \frac{2\sigma}{\sqrt{n(n-1)}} \left[\frac{n+1}{3} + \frac{2(n-2)\sqrt{3}}{\pi} - \frac{2(2n-3)}{\pi} \right]^{\frac{1}{2}} \quad \text{.....(36),}$$

$$\rightarrow \frac{2\sigma}{\sqrt{n}} \left[\frac{\pi - 6(2-\sqrt{3})}{3\pi} \right]^{\frac{1}{2}} = \cdot 8068 \frac{\sigma}{\sqrt{n}} \text{ as } n \text{ increases} \quad \text{.....(37).}$$

From (35), it is seen that

$$E_5 = \tfrac{1}{2}\sqrt{\pi}\, g^* \quad \text{.....(38)}$$

may be used as an unbiassed estimate† of σ, and will have for its standard error

$$\sigma_{E_5} = \frac{\sigma}{\sqrt{n(n-1)}} \left[\frac{\pi}{3}(n+1) + 2(n-2)\sqrt{3} - 2(2n-3) \right]^{\frac{1}{2}} \quad \text{.....(39),}$$

$$\rightarrow \frac{\sigma}{\sqrt{n}} \left[\frac{\pi - 6(2-\sqrt{3})}{3} \right]^{\frac{1}{2}} = \cdot 715 \frac{\sigma}{\sqrt{n}} \text{ as } n \rightarrow \infty \quad \text{.....(40).}$$

E. S. Pearson and O. L. Davies‡ have discussed the relative merits of four different estimates of σ in current use, and it is interesting to compare with these the estimate obtained above. The four estimates they consider are as follows:

$$(1)\ E_1 = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} \quad (\bar{x} \text{ is the sample mean}) \quad \text{.....(41),}$$

$$(2)\ E_2 = \frac{1}{b_n} \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}} \quad \text{.....(42),}$$

$$(3)\ E_3 = \frac{1}{d_n} w \quad (w \text{ is the range}) \quad \text{.....(43),}$$

$$(4)\ E_4 = \frac{1}{f_n} m \quad (m \text{ is the average error or mean deviation}) \quad \text{.....(44).}$$

* See equations (41)—(44) for estimates E_1 to E_4.

† I.e. an estimate whose mean value in repeated random samples equals the population value.

‡ "Methods of estimating from Samples the Population Standard Deviation," *Journal of Royal Statistical Society. Agricultural and Industrial Research Section Supplement*, Vol. I. (1934), pp. 76—89.

Here b_n, d_n and f_n are constants depending upon the size of the sample n, for which the authors gave certain tables. All these four estimates are unbiassed*. The following table giving their standard errors, in which the first four rows are taken from the paper by Pearson and Davies, provides a comparison of reliability.

TABLE II. *Standard Errors of Estimates of σ in terms of σ as unit.*

Estimate	Number of observations in the sample $= n$								
	2	3	4	5	6	10	15	20	50
E_1	·603	·463	·389	·341	·308	·232	·187	·161	·101
E_2	·756	·523	·422	·363	·323	·239	·191	·163	·101
E_3	·756	·525	·427	·371	·335	·259	·217	·195	·145
E_4	·756	·525	·430	·373	·334	·250	·201	·173	·108
E_5	·756	·525	·425	·366	·326	·241	·193	·165	·103

We see that in this case (i.e. for the normal distribution) for $n > 3$, E_5 has a smaller standard error than E_3 and E_4. As n increases the standard errors of E_5 and E_4 approach $\cdot 715\sigma/\sqrt{n}$ and $\cdot 756\sigma/\sqrt{n}$ respectively. Hence E_5 might, from the theoretical point of view, be preferred to E_4. But it must be remembered that the calculation of g (and E_5) involves first the arrangement of the data in order of magnitude, then the calculation of U from (2) and V from (3) and finally an application of the formula (1). E_4 however can be calculated from the formula

$$E_4 = \frac{2}{n}[\text{sum of observations greater than } \bar{x} - k\bar{x}] \times \frac{1}{f_n} \quad \ldots\ldots\ldots(45),$$

where $\bar{x}$ is the arithmetic mean and k is the number of observations greater than $\bar{x}$. To the computer, who has a table of f_n available, this greater simplicity in the calculation of E_4 would probably outweigh the slightly greater accuracy of E_5.

(ii) *Exponential distribution* [equation (33)].

Substituting the values of the integrals from the third column of Table I, into (9) and (29)—(31), we find

$$\bar{g} = \sigma \quad \ldots\ldots\ldots(46),$$

$$\sigma_g = \sqrt{\frac{2(2n-1)}{3n(n-1)}}\,\sigma \quad \ldots\ldots\ldots(47),$$

$$\to \frac{2}{\sqrt{3n}}\,\sigma \text{ as } n \to \infty \quad \ldots\ldots\ldots(48).$$

g is therefore an unbiassed estimate of σ. Another estimate of σ is based on the distance between the sample mean ($\bar{x}$) and the lowest observation in the sample (x_1) or $\bar{x} - x_1 = d$, say. It is known that the sampling distribution of d is†

$$p(d) = \left(\frac{n}{\sigma}\right)^{n-1} \frac{1}{(n-2)!}\, d^{n-2} e^{-nd/\sigma} \quad \ldots\ldots\ldots(49).$$

* Actually E_1^2 is an unbiassed estimate of σ^2.

† J. Neyman and E. S. Pearson, *Biometrika*, Vol. XXA. (1928). p. 223.

Hence the expectation of d is $\sigma(n-1)/n$ and its standard error is $\sigma\sqrt{n-1}/n$. Therefore $nd/(n-1)$ is an unbiassed estimate of σ and

$$\frac{\text{Standard error of } g}{\text{Standard error of } nd/(n-1)} = \sqrt{\frac{2(2n-1)}{3n}} \quad \text{.....(50),}$$

$$\to \frac{2}{\sqrt{3}} \text{ as } n \to \infty \quad \text{.....(51).}$$

Thus g has a larger standard error than d and consequently is the less accurate estimate of σ.

(iii) *Rectangular distribution* [equation (34)].

Substituting the values of the integrals from the fourth column of Table I into (9) and (29)—(31), we find

$$\bar{g} = \tfrac{1}{3}k \quad \text{.....(52),}$$

$$\sigma_g = \frac{k}{3}\sqrt{\frac{n+3}{5n(n-1)}} \quad \text{.....(53)}$$

$$= \frac{k}{3n\sqrt{5}} \text{ as } n \to \infty \quad \text{.....(54).}$$

$3g$ is thus an unbiassed estimate of k. If w is the sample range, or distance between the highest and lowest observations, then

$$\bar{w} = \frac{n-1}{n+1}k \quad \text{.....(55),}$$

and

$$\sigma_w = \frac{k}{n+1}\sqrt{\frac{2(n-1)}{n+2}}^{*} \quad \text{.....(56).}$$

Hence $\dfrac{n+1}{n-1}w$ is an unbiassed estimate of k having standard error

$$k\sqrt{\frac{2}{(n-1)(n+2)}}.$$

We see that

$$\frac{\text{Standard error of } 3g}{\text{Standard error of } \dfrac{n+1}{n-1}w} = \sqrt{\frac{(n+3)(n+2)}{10n}} \quad \text{.....(57),}$$

$$> 1 \text{ for } n > 3.$$

From (57) it is seen that the estimate based on range is in general a better estimate of k, and therefore of the population range, than is $3g$.

4. *Conclusion.*

In the case of the exponential and rectangular distributions the Mean Difference as an estimate of the scale of variation in the population seems to have no advantage over the simpler estimates with which it has been compared. For the normal distribution, $\frac{1}{2}\sqrt{\pi} \times$ Mean Difference may be used to estimate the population standard deviation; it is less reliable than estimates based on squares of deviations from the mean, but somewhat more reliable than that based on the Mean Deviation and considerably more reliable (for $n > 10$, say) than that based on the Range.

In conclusion, I would like to thank Professor Pearson for suggesting this problem and giving me help and guidance.

* J. Neyman and E. S. Pearson, *loc. cit.* p. 217.

MISCELLANEA.

(i) Fiducial Limits for the Poisson Distribution.

By F. GARWOOD, Ph.D.

1. *The Method of Fiducial or Confidence Limits.*

A situation of very common occurrence in statistics arises when a random sample is drawn from a population which is not completely specified, and it is desired to draw some inference about the population on the basis of the sample. It is usual to assume, from independent evidence, a mathematical form for the chance distribution of the variate, so that it will be completely specified if the value of one or more parameters is known. Thus past experience may tell us that the population of heights of men belonging to a homogeneous race group is represented by the familiar normal distribution, of which the parameters necessary to specify it are the mean and standard deviation. Any inference which can be made about the parameters on the basis of the sample is of course subject to error, since two different populations can give rise to the same sample. It is the fundamental property of the method of fiducial limits*, however, that the risk of making incorrect inferences by this method can be controlled. The value of the method is therefore that if a large number of situations arise in which the rules of the method are applied, one can have confidence that, in the long run, the proportion of inferences which are false does not exceed appreciably a pre-determined ratio, say 5%, or, if more stringency be required, 1% or ·01%.

2. *Application of the Method to the Poisson Distribution*†.

The Poisson limit to the binomial defines a hypothetical population containing all the positive integers, including zero, and the proportion of times the integer x occurs is

$$\frac{e^{-m}m^x}{x!} \qquad \text{.....(1),}$$

where m is the mean, the only parameter necessary to define the distribution.

Suppose we are confronted by a physical phenomenon which is known to give rise to a Poisson distribution, such as the occurrence of random and independent events in time or space, and we wish to infer something about the mean on the data of one observation, which is an integer. Suppose further that we do not wish the risk of error to rise above a probability of ·05, and that we are only interested in inferring a lower limit above which the mean lies. The rule to be followed is that we look up the entry in the 5% "lower limit" column of the Table of fiducial limits, given below, corresponding to the sample integer observed (or interpolate into the table if necessary), and make the statement that the mean lies above this. As stated above, this rule, which may be applied in sampling from a series of different distributions, will lead to false statements being made on a proportion of occasions which is never much more than once in 20 in the long run, though of course one cannot tell on *which* particular occasions the rule has failed.

* R. A. Fisher, *Proc. Camb. Phil. Soc.* xxvi. (1930), p. 528; *Proc. Roy. Soc.* A, cxxxix. (1933), p. 343. See also J. Neyman, *Journ. R. Statistical Soc.* xcvii. (1934), p. 589. Neyman uses the term "confidence limit."

† The reader is referred to a paper entitled "Statistical principles of routine work in testing clover seeds for dodder" by J. Przyborowski and H. Wileński, *Biometrika*, xxvii. (1935), pp. 273—292, dealing with somewhat the same problem. [Dr Garwood's paper was part of his thesis for the Ph.D. degree of London University completed in June 1934. Its publication has unfortunately been delayed overlong. Ed.]

Similar results hold for the 1 °/₀ lower limit, for the upper limit, and for the upper and lower limits combined; in the last cases, the probabilities of error are, respectively, not greater than ·10 and ·02.

This is an example of the application of the method of fiducial limits to a distribution in which the variate can only assume discrete values; it is important to note that in such cases the table of fiducial limits can only be calculated so that the probability of error will not exceed certain values, whereas in the case of continuous distributions it can be fixed exactly at a desired set of values. The consideration of the actual value of the probability of error is returned to later.

3. *Theory of the Method.*

From equation (1) it follows that the probability of the occurrence of either 0, 1, 2, ... or x as one random sample from the Poisson distribution with mean x is

$$P\{x \mid m\} = e^{-m}\left(1+\frac{m}{1!}+\frac{m^2}{2!}+\ldots\ldots+\frac{m^x}{x!}\right) \quad \ldots\ldots(2).$$

By differentiating this with respect to m, it is seen that

$$P\{x \mid m\} = \int_m^\infty \frac{e^{-t}t^x}{x!}\,dt \quad \ldots\ldots(3),$$

$$= 1 - \int_0^m \frac{e^{-t}t^x}{x!}\,dt \quad \ldots\ldots(4).$$

Corresponding to values of x from 0 to 50 we have calculated the values of m for which

$$P\{x-1 \mid m\} = \cdot 99 \text{ and } \cdot 95 \quad \ldots\ldots(5),$$

and

$$P\{x \mid m\} = \cdot 05 \text{ and } \cdot 01 \quad \ldots\ldots(6).$$

These are the fiducial limits for m corresponding to x, which we have denoted by $m_{\cdot 99}(x)$, $m_{\cdot 95}(x)$, $m_{\cdot 05}(x)$, and $m_{\cdot 01}(x)$ respectively, and they are given in Table I*. The lower limits corresponding to $x=0$ are defined as zero.

To demonstrate the fundamental property of the fiducial limits, consider a Poisson distribution with mean $m=15$, say, and the upper 1 °/₀ limit for m corresponding to samples drawn from it. There is an integer s such that

$$m_{\cdot 01}(s) \leqq m = 15 < m_{\cdot 01}(s+1) \quad \ldots\ldots(7),$$

and from the column for $m_{\cdot 01}(x)$ we see that $s=6$. If now the statements "$m < m_{\cdot 01}(x)$" are made according as the various values of x arise in random sampling, a false statement will be made when x is either 0, 1, 2, 3, 4, 5 or 6 $(=s)$, since the statement will then be one of the following: "$m<4{\cdot}61$," "$m<6{\cdot}64$," ..... "$m<14{\cdot}57$," all of which are false in this case. The probability of one of these events occurring is, by definition,

$$P\{s \mid m\} = P\{6 \mid 15\};$$

this function decreases with increasing m, and since $m \geqq m_{\cdot 01}(s)$, we must have

$$P\{6 \mid 15\} = P\{s \mid m\} \leqq P\{s \mid m_{\cdot 01}(s)\} = \cdot 01 \quad \ldots\ldots(8),$$

by definition of $m_{\cdot 01}(s)$, (see equation (6)). It follows that if the above rule is observed in sampling from any Poisson distribution, i.e. if the statement

$$m < m_{\cdot 01}(x) \quad \ldots\ldots(9)$$

is made according to whatever integer x arises, then the probability of a false statement is

$$P\{s \mid m\} \equiv P(m) \quad \ldots\ldots(10),$$

* The upper limits correspond to those given by Przyborowski and Wileński in their Table V (*loc. cit.* p. 288). These authors, however, only give the limit to 1 place of decimals and have not tabled any lower limits.

where s is given by

$$m_{\cdot 01}(s) \leqq m < m_{\cdot 01}(s+1) \quad \ldots\ldots\ldots\

The values of the functions $P(m)$, $P'(m)$, and of their sum $P''(m)$, are shown in Figs. 1 and 2 for two ranges of m. $P(m)$ and $P'(m)$ are discontinuous at those values of m which are respectively upper and lower 1 °/₀ fiducial limits (i.e. the values of m given under $m_{\cdot 01}(x)$ and $m_{\cdot 99}(x)$ of Table I), and the functions reach up to ·01 at these points. $P''(m)$ is discontinuous at both these sets of points, but, over the range investigated, it is always less than ·02. Up to $m=4{\cdot}61$, $P(m)$ is zero, and hence $P''(m)=P'(m)$. Fig. 1 gives the graph of $P'(m)$ for a small range of m, starting from zero, and shows the first three branches of the function. In Fig. 2 the three functions are plotted from $m=4{\cdot}0$ to $10{\cdot}6$ on a larger horizontal scale.

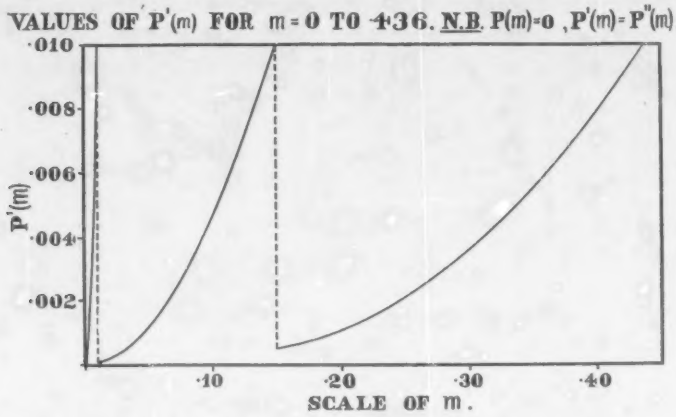

Fig. 1.

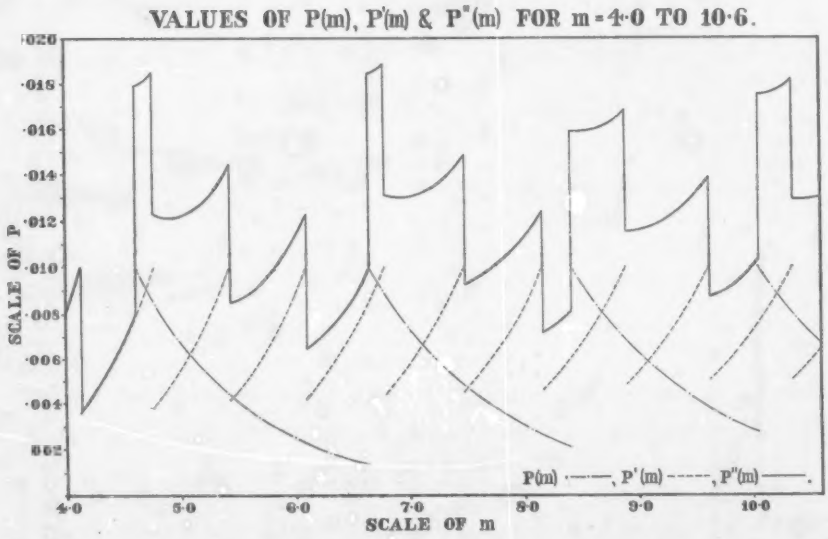

Fig. 2.

In the application of the fiducial method, the values of m are never known, so that the values recorded in the figure cannot be used exactly. If an *a priori* distribution of means could be assumed, then by integration of $P(m)$, $P'(m)$ and $P''(m)$ one could calculate the absolute risk of error in the fiducial statements, and the result would be less than ·01, ·02, ·05, ·10, as the case may be. This situation rarely arises in practice, but Fig. 2 shows that the limits given to the probability of error are very conservative ones and the true probability may often be considerably less.

4. *Method of Calculation of Table I.*

For values of x up to 14, the lower limits were calculated from Fisher's χ^2 Table* by means of the transformations:

$$x=\frac{n-2}{2},\ m=\tfrac{1}{2}\chi^2,\quad (P=\cdot 99 \text{ and } \cdot 95) \qquad \text{.....(16).}$$

For values of x up to 15, the upper limits were calculated from the same table with

$$x=\frac{n}{2},\ m=\tfrac{1}{2}\chi^2,\quad (P=\cdot 01 \text{ and } \cdot 05) \qquad \text{.....(17).}$$

This can be done because the distribution of χ^2, with n degrees of freedom, has a probability integral, given in the table, equal to

$$P=\int_{\frac{1}{2}\chi^2}^{\infty} \frac{e^{-\frac{\chi^2}{2}}\left(\frac{\chi^2}{2}\right)^{\frac{n-2}{2}}}{\frac{n-2}{2}!}\, d\left(\frac{\chi^2}{2}\right) \qquad \text{.....(18)}$$

which reduces to (3) after the necessary transformations have been applied.

For the remainder of the table, inverse interpolation was made into the *Tables of the Incomplete* Γ*-Function*†, which is a table of the function

$$I(u,p)=\int_0^{u\sqrt{p+1}} \frac{e^{-t}t^p}{p!}\, dt = 1-P(p\,|\,u\sqrt{p+1}) \qquad \text{.....(19).}$$

5. *Approximate Formulae for Large Samples.*

The Γ-Function Table stops at $p=50$, so that approximations must be used to calculate the fiducial limits for m corresponding to values of x greater than 50. At the foot of his χ^2 Table, Fisher suggests that for $n>30$, $\sqrt{2\chi^2}-\sqrt{2n-1}$ may be used as a normal variate with unit standard deviation. A more accurate formula, though less simple, has been given by Wilson and Hilferty‡. This assumes that $\left(\frac{\chi^2}{n}\right)^{\frac{1}{3}}$ is normally distributed about $1-\frac{2}{9n}$ with standard deviation equal to $\sqrt{\frac{2}{9n}}$. Fiducial limits for $m=\frac{1}{2}\chi^2$ have been calculated from these approximations for $x=20$, 30, 40, 50, and compared in Table II with the true values obtained from the Γ-Function Table. The error in the limits based on Fisher's approximation only appears to decrease with increasing x in one instance, whereas the error in Wilson's and Hilferty's formula decreases in all cases and is much smaller.

By using the equations (16) and (17) this table also serves as a comparison of the approximations for the significance levels of χ^2 for large values of n, the number of degrees of freedom.

* R. A. Fisher, *Statistical Methods for Research Workers*, Table III.

† Edited by Karl Pearson (1922).

‡ *Nat. Acad. Sci.* XVII. No. 12 (1931), p. 684.

TABLE II.

Comparison of Approximate Formulae for Fiducial Limits of m for Large Values of x.

m_T = True value, obtained from Γ-Function Tables.
m_F = Approximate value, obtained from Fisher's formula.
m_W = Approximate value, obtained from Wilson's and Hilferty's formula.

		x	m_T	m_F	$m_T - m_F$	m_W	$m_T - m_W$
Lower Limits	1 %	20	11·082	10·764	·318	11·070	·012
		30	18·742	18·414	·328	18·732	·010
		40	26·770	26·436	·334	26·761	·009
		50	35·032	34·694	·338	35·025	·007
	5 %	20	13·255	13·116	·139	13·254	·001
		30	21·594	21·455	·139	21·594	·000
		40	30·196	30·056	·140	30·196	·000
		50	38·965	38·825	·140	38·965	·000
Upper Limits	1 %	20	33·103	32·700	·403	33·113	−·010
		30	45·401	45·003	·398	45·409	−·008
		40	57·347	56·953	·394	57·355	−·008
		50	69·067	68·676	·391	69·074	−·007
	5 %	20	29·062	28·919	·143	29·060	·002
		30	40·691	40·548	·143	40·689	·002
		40	52·069	51·926	·143	52·068	·001
		50	63·287	63·144	·143	63·286	·001

(ii) Note on Karl Pearson's Paper: "On a method of ascertaining limits to the actual number of marked members in a population of given size from a sample." [*Biometrika*, Vol. xx^A. pp. 149—174.]

By K. RAGHAVAN NAIR, M.A.

Madras University.

On page 151 of this paper Professor Karl Pearson has proved at some length the following result:

$$1+\frac{r+1}{1}\frac{N-n}{N-r}+\frac{(r+1)(r+2)}{1.2}\frac{(N-n)(N-n-1)}{(N-r)(N-r-1)}+\ldots+\frac{\lfloor s}{\lfloor r}\frac{\lfloor N-s}{\lfloor N-r}=\frac{\lfloor N+1}{\lfloor n+1}\frac{\lfloor n-r}{\lfloor N-r}\quad\ldots\ldots(1),$$

where $r+s=n$ and all the letters denote positive integers.

The problem discussed in this paper is the frequency distribution of populations of size N with number of marked individuals varying from $r, r+1, \ldots$ to $N-s$, obtained on proceeding by the method of Inverse Probability from the knowledge that a sample of n contains r marked and s unmarked individuals. The terms on the left-hand side of (1) are respectively proportional to the probabilities of populations of size N containing $r, r+1, r+2, \ldots N-s$ marked individuals. It is clear, therefore, that there are $N-r-s+1$ or $N-n+1$ terms in the above series.

Professor Pearson derives the above result (1) from the sum of the hypergeometric series

$$1+\frac{\alpha\beta}{\lfloor 1\,\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2\,\gamma(\gamma+1)}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\lfloor 3\,\gamma(\gamma+1)(\gamma+2)}+\ldots=\frac{\Gamma(\gamma-\alpha-\beta)\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \quad \ldots\ldots(2),$$

by appropriate substitutions in (2) for α, β and γ, and after certain transformations.

He remarks that his proof "may strike the reader as artificial, and may raise doubts in his mind" (*loc. cit.* p. 152), and gives a numerical test to convince the reader of the accuracy of the result obtained.

An alternative proof based on simple binomial expansions is given below.

Consider the identities,

$$(1-x)^{-r-1}\equiv 1+\frac{r+1}{1}x+\frac{(r+1)(r+2)}{1.2}x^2+\ldots+\frac{(r+1)(r+2)\ldots(r+N-n)}{\lfloor N-n}x^{N-n}+\ldots \quad \ldots\ldots(3),$$

$$(1-x)^{-s-1}\equiv 1+\frac{s+1}{1}x+\frac{(s+1)(s+2)}{1.2}x^2+\ldots+\frac{(s+1)(s+2)\ldots(s+N-n)}{\lfloor N-n}x^{N-n}+\ldots \quad \ldots\ldots(4),$$

$$[0<x<1]$$

where, let us assume, r and s are positive integers and $r+s=n$.

Multiplying together the sides of (3) and (4), we have

$$\left[1+\frac{\lfloor r+1}{\lfloor r\,\lfloor 1}x+\frac{\lfloor r+2}{\lfloor r\,\lfloor 2}x^2+\ldots+\frac{\lfloor N-s}{\lfloor r\,\lfloor N-n}x^{N-n}+\ldots\right]$$
$$\times\left[1+\frac{\lfloor s+1}{\lfloor s\,\lfloor 1}x+\frac{\lfloor s+2}{\lfloor s\,\lfloor 2}x^2+\ldots+\frac{\lfloor N-r}{\lfloor s\,\lfloor N-n}x^{N-n}+\ldots\right]$$
$$\equiv(1-x)^{-r-s-2}\equiv(1-x)^{-n-2} \quad \ldots\ldots(5).$$

This being identically true, coefficients of like terms must be equal on both sides. Consider the terms containing x^{N-n}.

The coefficient of x^{N-n} on the left-hand side of (5) is

$$\frac{\lfloor N-r}{\lfloor s\,\lfloor N-n}+\frac{r+1}{1}\frac{\lfloor N-r-1}{\lfloor s\,\lfloor N-n-1}+\frac{(r+1)(r+2)}{1.2}\frac{\lfloor N-r-2}{\lfloor s\,\lfloor N-n-2}+\ldots$$
$$+\frac{s+1}{1}\frac{\lfloor N-s-1}{\lfloor r\,\lfloor N-n-1}+\frac{\lfloor N-s}{\lfloor r\,\lfloor N-n}$$
$$=\frac{\lfloor N-r}{\lfloor s\,\lfloor N-n}\left[1+\frac{r+1}{1}\frac{N-n}{N-r}+\frac{(r+1)(r+2)}{1.2}\frac{(N-n)(N-n-1)}{(N-r)(N-r-1)}+\ldots+\frac{\lfloor s\,\lfloor N-s}{\lfloor r\,\lfloor N-r}\right] \quad \ldots\ldots(6).$$

The coefficient of x^{N-n} on the right-hand side of (5) is

$$\frac{(n+2)(n+3)\ldots(N+1)}{\lfloor N-n}=\frac{\lfloor N+1}{\lfloor n+1\,\lfloor N-n} \quad \ldots\ldots(7).$$

Equating (6) and (7) and dividing throughout by $\dfrac{\lfloor N-r}{\lfloor s\,\lfloor N-n}$, we get

$$1+\frac{r+1}{1}\frac{N-n}{N-r}+\frac{(r+1)(r+2)}{1.2}\frac{(N-n)(N-n-1)}{(N-r)(N-r-1)}+\ldots+\frac{\lfloor s}{\lfloor r}\frac{\lfloor N-s}{\lfloor N-r}$$
$$=\frac{\lfloor N+1}{\lfloor n+1\,\lfloor N-n}\frac{\lfloor s\,\lfloor N-n}{\lfloor N-r}=\frac{\lfloor N+1}{\lfloor n+1}\frac{\lfloor n-r}{\lfloor N-r},$$

since $s=n-r$. This is Karl Pearson's result given in (1).

It should be noted that in his equation (xiii) (*loc. cit.* p. 153) Professor K. Pearson wrote for the mean of the number of marked individuals in the population $(r+1)(N-n)/(n+2)$ instead of $r+(r+1)(N-n)/(n+2)$. A similar omission of r occurs in equation (xiv) for the mode. The correct expressions were however used in the numerical example given a few lines below. In the formulae the origin has been taken at r, the start of the distribution, instead of at zero.

(iii) **J. Arthur Harris, Botanist and Biometrician.** Edited by C. Otto Rosendahl, Ross Aiken Gortner and George O. Burr. Published by the University of Minnesota Press, Minneapolis, and Humphrey Milford, Oxford University Press, London, 1936. Price 11*s.* 6*d.*

Dr J. A. Harris, the subject of this volume, died in 1930 at the early age of 50 years; he had led a strenuous life of independence since the age of 13, and one can only marvel at the achievements crowded into his short life. The volume consists of three sections under the headings of *Harris the Man*, *Harris the Botanist* and *Harris the Biometrician* respectively.

The first section draws an interesting picture of Harris from his childhood, when he was already showing signs of virility and of an enthusiasm capable of overriding difficulties which beset his path. Those of us who knew him chiefly as a Biometrician, in later years, will be interested in reading how early he became impressed with the need for exact quantitative data in biology, and was urging the importance of applying statistical methods to the interpretation of biological facts. Harris was a pioneer in the application of Biometry to his own subject of Botany and, having learned the value of his tools, he proceeded to make use of them in a number of problems, concerned very often with vital statistics; he was evidently ready to apply himself to any problem brought to him, but his primary concern was for the study of Botany and its many practical applications.

Readers of *Biometrika* will read with sympathetic interest the account of Harris's two winters, of 1908–9 and 1909–10, spent in Karl Pearson's laboratory in London; eleven years later, on being awarded the Weldon Medal, Harris wrote to Karl Pearson—"For me the years have gone by with intense and unremitting toil, but I want to say this—there has not been a week, and scarcely a day, that I have not thought of Gower Street and Hampstead Heath, and of the enthusiasm with which the little group of us worked in the old Laboratory....My work has been an uphill fight from the beginning, but I can honestly say that I have never for a moment left the guns, turned back because of difficulties or opposition, or lost sight of the ideals of reasonableness, logic and precision that a few of us have learned in your laboratory... and I could not accept this distinction without assuring you of the fullest realisation of the debt that I owe to your influence during the past 20 years." This letter is characteristic of the man portrayed throughout the volume; it testifies to the energy and unfailing zeal further illustrated by the Bibliography of his published papers, given on pages 199—209, and including some 330 entries.

Harris was surely of an energetic stock; we are told that round his baby plate were the words "Dost thou love life? Then do not squander time. There will be sleeping enough in the grave"!

In the preface to the volume J. B. Johnstone writes of Harris's ability as an administrator and says that "no book can convey an adequate impression of a man so virile, so many sided and so human"—yet the authors have succeeded in portraying an arresting personality with the qualities of a pioneer.

J. B.

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